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Smooth Bifurcation of Symmetric Periodic Solutions of Functional Differential Equations

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Many mathematical models in biology and physics lead to functional differential equations, e.g.,

—the equation $\dot{n}(t) = \lambda n(t)(1 - n(t-\alpha)/N)$, $\lambda > 0$, $N > 0$, $\alpha > 0$ for the growth of populations (Hutchinson [8]),

—the equation $\dot{x}(t) = [cx(t-\alpha)]^8 e^{-x(t-\alpha)} - sx(t)$ for the density of red blood cells (Mackey and Glass [11]; Lasota and Wazeswka–Czyzewska [10]),

—the equation $\dot{x}(t) = \delta - \sin(x(t-\alpha) + \omega)$ for phase locked loops (Furumochi [5]).

In this paper we study the differential delay equation

$$\dot{x}(t) = f(x(t-\alpha)) \qquad \text{or equivalently} \qquad \dot{x}(t) = \alpha f(x(t-1)) \qquad (\alpha f)$$

with odd $f \in C(\mathbb{R})$.

In contrast to the one-dimensional ordinary differential equation this equation can have periodic solutions as the example $\dot{x}(t) = -\pi/2\, x(t-1)$ with solutions $x(t) = z \sin(\pi/2\, t)$, $z > 0$ shows. Numerical bifurcation diagrams (e.g., [6, 14]) indicate a complicated structure of the period solutions depending on the delay parameter α. One finds various intersecting smooth curves. One task of mathematics is to prove the existence of periodic solutions. In this paper we are mainly interested in *symmetric* periodic solutions, i.e., solutions x of (αf) which fulfil the symmetry property

$$x(\cdot + \tau) = -x, \qquad \tau > 0. \qquad (S)$$

From now on let $f \in C^1(\mathbb{R})$ be odd, $xf(x) < 0$ for $x \neq 0$, $f(0) = 0$, and $f'(0) < 0$. In this case a paper of Kaplan and Yorke [9] yields for every amplitude $z > 0$ an $\alpha(z) \in \mathbb{R}$ and exactly one symmetric periodic solution

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$x(\cdot, z)\colon \mathbb{R} \to \mathbb{R}$ of $(\alpha(z) f)$ such that $x(\cdot + 2, z) = -x(\cdot, z)$ (cf. Theorem 1.1). Moreover, $\mathbb{R} \times \mathbb{R}^+ \ni (t, z) \mapsto x(t, z) \in \mathbb{R}$ is differentiable and $\lim_{z \to 0^+} \alpha(z) = -\pi/2f'(0)$; i.e., the Kaplan–Yorke solutions form a branch of symmetric periodic solutions which bifurcates at $\alpha = -\pi/2f'(0)$ from the trivial solutions $x \equiv 0$ and which can be parameterized by the amplitude in a differentiable way. We call this branch the *primary branch* of f.

Walther [16] proved in 1983 the existence of secondary bifurcation for, e.g., $f = -\sin$. His result can be stated as follows (Theorem 6.2 in [16]).

There is a point on the primary branch of f *such that every neighbourhood of this point contains a periodic solution, which is* not *symmetric* (*and hence lies not on the primary branch*) (see Fig. 1).

He does not prove, that the bifurcating solutions are lying on a smooth curve, which is to be expected from numerical studies.

In the present paper we give conditions for the existence of a bifurcation point from which a branch of symmetric periodic solutions bifurcates (Theorem 3.1), and proof that $f(x) = -x/(1 + x^2)$, $x \in \mathbb{R}$ satisfies these conditions. Moreover we will show, that the bifurcating solutions are lying on a smooth (secondary) branch. Hence we say that we have found a *smooth bifurcating point* on the primary branch (see Fig. 2).

Let us describe the organization of this paper. The idea is to apply bifurcation from a simple eigenvalue. To this end we transform our problem in Section 1 into a fixed point problem. This is done by defining an appropriate semi-flow and a Poincaré map. Next we prove a version of the theorem on the bifurcation from a simple eigenvalue (Theorem 1.2) that fits to our situation. The main result of Section 1 is Theorem 1.3. There an operator $U(z)$ appears, which depends on the amplitude z. With $U(z)$ the

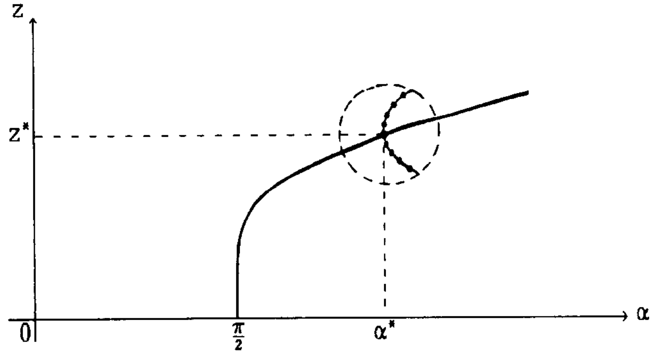

FIGURE 1

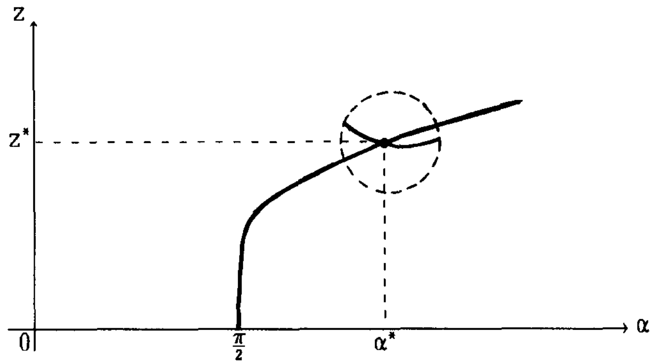

FIGURE 2

conditions of the theorem on the bifurcation from a simple eigenvalue can be formulated as follows:

(i) $\dim \ker(U(z)+1)=1$,

(ii) $U'(z)\,\xi \notin \operatorname{im}(U(z)+1)$ for some $\xi \in \ker(U(z)+1)$, $\xi \neq 0$,

(iii) $U(z)+1$ is a Fredholm operator with index 0.

Hence we first have to study the problem $-1 \in \sigma(U(z))$. To this end we transform the eigenvalue problem into a boundary value problem for a linear two-dimensional system of ordinary differential equations (Lemma 2.3). The study of the fundamental solution of this system yields a condition for $-1 \in \sigma(U(z))$ (Theorem 2.1). Moreover we find

$$\ker(U(z)+1) \neq \{0\} \Rightarrow \dim \ker(U(z)+1)=1.$$

In Section 3 we finally study the conditions (ii) and (iii) and summarize our results in Theorem 3.1. The conditions for the existence of a smooth bifurcation point on the primary branch of f given in Theorem 3.1 now contain only the Kaplan–Yorke solutions and can easily be checked numerically. Moreover one can find conditions for f from which these conditions can be derived. However for $f(x) = -x/(1+x^2)$, $x \in \mathbb{R}$ it is possible to prove the assumptions of Theorem 3.1 directly. This is done in Section 4 where we add some supplementary remarks.

1. PRELIMINARIES

We need some statements on differential delay equations. Proofs can be found, e.g., in [7].

Let $f \in C(\mathbb{R})$, $\alpha \in \mathbb{R}$ and equip the Banach space $C := C([0, 1])$ with the sup-norm.

LEMMA 1.1. *For $\varphi \in C$ there is exactly one continuous map $x: \mathbb{R}_0^+ \to \mathbb{R}$ such that*:

(1) $x|_{[0,1]} = \varphi$,

(2) *x is differentiable on $]1, \infty[$ and the right-hand side derivative exists at $t = 1$.*

(3) *$\dot{x}(t) = \alpha f(x(t-1))$ for all $t \geqslant 1$.*

We denote this unique solution by $x(\cdot, \alpha, \varphi)$. It is obtained by piecewise integration, for example, for $t \in [1, 2]$ we have

$$x(t, \alpha, \varphi) = \varphi(1) + \alpha \int_1^t f(\varphi(s-1))\, ds.$$

Remark 1.1. If $f \in C^k(\mathbb{R})$, then $x(\cdot, \alpha, \varphi)$ is a C^j-map on $[j, \infty[$ for $j = 0, ..., k+1$, and if $x(\cdot, \alpha, \varphi)$ is periodic, $x(\cdot, \alpha, \varphi)$ is of class C^{k+1} on $\mathbb{R}_0^+$, in particular $\varphi \in C^{k+1}([0, 1])$.

DEFINITION 1.1. For $t \in \mathbb{R}_0^+$, $\alpha \in \mathbb{R}$, and $\varphi \in C$ let $X(t, \alpha, \varphi) := x(t + \cdot, \alpha, \varphi)|_{[0,1]} \in C$.

This yields a map $X: \mathbb{R}_0^+ \times \mathbb{R} \times C \to C$, and we have

LEMMA 1.2. *Let $f \in C^k(\mathbb{R})$, $k \in \mathbb{N}_0$.*

(i) *X is continuous and k-times continuously differentiable on $]k, \infty[\times \mathbb{R} \times C$.*

(ii) *The partial derivatives $X_\alpha: \mathbb{R}_0^+ \times \mathbb{R} \times C \ni (t, \alpha, \varphi) \mapsto DX(t, \cdot, \varphi)(\alpha) \in C \cong L(\mathbb{R}, C)$ and $X_\varphi: \mathbb{R}_0^+ \times \mathbb{R} \times C \ni (t, \alpha, \varphi) \mapsto DX(t, \alpha, \cdot)(\varphi) \in L(C, C)$ are continuous, and continuously differentiable on $]1, \infty[\times \mathbb{R} \times C$.*

(iii) *For all $\alpha \in \mathbb{R}$ $X(\cdot, \alpha, \cdot): \mathbb{R}_0^+ \times C \to C$ is a semi-flow on C.*

Next we define the variational equation for (αf):

DEFINITION 1.2. Let $f \in C^1(\mathbb{R})$, $\alpha \in \mathbb{R}$, $\varphi \in C$, and $\xi \in C$. Let $w(\cdot, \alpha, \varphi, \xi): \mathbb{R}_0^+ \to \mathbb{R}$ denote the unique solution of the linear initial-value delay problem

$$\dot{w}(t) = \alpha f'(x(t-1, \alpha, \varphi))\, w(t-1), \qquad t \geqslant 1,\ w|_{[0,1]} = \xi.$$

Furthermore let $W(t, \alpha, \varphi) := (C \ni \xi \mapsto w(t + \cdot, \alpha, \varphi, \xi)|_{[0,1]} \in C) \in L(C, C)$ for $t \geqslant 0$.

Again $w(\cdot, \alpha, \varphi, \xi)$ is obtained by piecewise integration. If $\varphi \in C$ is differentiable, then $w(t, \alpha, \varphi, \dot{\varphi}) = \dot{x}(t, \alpha, \varphi)$ for $t \geqslant 0$. Moreover

LEMMA 1.3. $X_\varphi = W$.

Now we transform the problem of finding periodic solutions into a fixed point problem. As for ordinary differential equations, this is done by defining an appropriate Poincaré map. Since we are only interested in periodic solutions which oscillate about $0 \in \mathbb{R}$, we assume $x(0) = 0$ or $x|_{[0,1]} \in C_0$, where $C_0 := \{\varphi \in C \mid \varphi(0) = 0\} \subseteq C$ is a closed subspace of C with codimension 1. Let $\Lambda \subseteq \mathbb{R}^+ \times C_0$ be the set of all $(\alpha, \varphi) \in \mathbb{R}^+ \times C_0$ such that:

$$\text{There is a } t > 1 \text{ with } x(t, \alpha, \varphi) = 0, \qquad \dot{x}(t, \alpha, \varphi) \neq 0$$
$$\text{and } x(s, \alpha, \varphi) \neq 0 \qquad \text{for} \quad s \in [1, t+1] \setminus \{t\}.$$

Then $\tau(\alpha, \varphi) := \inf\{t \geqslant 1 \mid x(t, \alpha, \varphi) = 0\}$ is defined for all $(\alpha, \varphi) \in \Lambda$ and

$$\tau(\alpha, \varphi) > 1, \qquad x(\tau(\alpha, \varphi), \alpha, \varphi) = 0, \qquad \dot{x}(\tau(\alpha, \varphi), \alpha, \varphi) \neq 0,$$
$$x(t, \alpha, \varphi) \neq 0 \qquad \text{for} \quad 1 \leqslant t < \tau(\alpha, \varphi) \qquad \text{and} \qquad \tau(\alpha, \varphi) < t \leqslant \tau(\alpha, \varphi) + 1.$$

The definition of Λ and the implicit function theorem yield

LEMMA 1.4. *Λ is an open subset of $\mathbb{R}^+ \times C_0$ and $\tau \in C^1(\Lambda)$.*

Now we are able to define our Poincaré map:

DEFINITION 1.3. We call $Q := (\Lambda \ni (\alpha, \varphi) \mapsto X(\tau(\alpha, \varphi), \alpha, \varphi) \in C_0)$ the "*semi*" *Poincaré map* (of f) and $P := (\Omega \ni (\alpha, \varphi) \mapsto Q(\alpha, Q(\alpha, \varphi)) \in C_0$ with $\Omega := \{(\alpha, \varphi) \in \Lambda \mid (\alpha, Q(\alpha, \varphi)) \in \Lambda\}$ is called the *Poincaré map* (of f) (see Fig. 3).

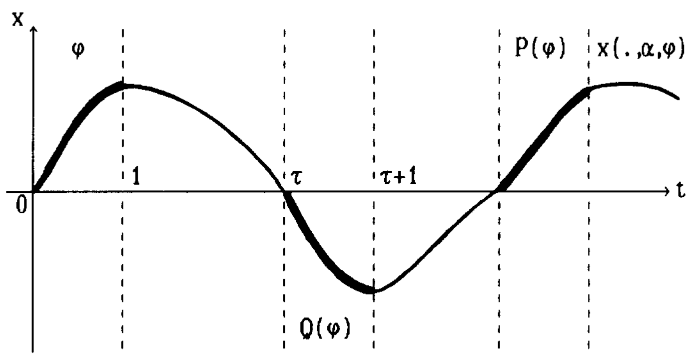

FIGURE 3

Since $Q(\alpha, \varphi) = X(\tau(\alpha, \varphi), \alpha, \varphi)$ for $(\alpha, \varphi) \in \Lambda$, it follows that Q is a C^1-map. Furthermore, $\Omega \subseteq \mathbb{R}^+ \times C_0$ is open, and P is a C^1-map too.

Now suppose $P(\alpha, \varphi) = \varphi$ for some $(\alpha, \varphi) \in \Omega$. Then $\varphi = X(\omega, \alpha, \varphi)$ where $\omega := \tau(\alpha, \varphi) + \tau(\alpha, Q(\alpha, \varphi))$, and hence $x(\cdot, \alpha, \varphi)$ is periodic with period ω. Thus we have found the fixed point equation $P(\alpha, \varphi) = \varphi$ for periodic solutions. Next we give an analogous equation for symmetric periodic solutions. For that purpose we need the following:

Definition 1.4. Let $f \in C(\mathbb{R})$ and $\alpha > 0$. A continuous map $x: \mathbb{R}_0^+ \to \mathbb{R}$ is called a *symmetric periodic solution* of (αf), if

(1) x is a solution of (αf) with $x(0) = 0$,

(2) there is a $\tau > 0$ with

  (i) $x(\cdot + \tau) = -x$,

  (ii) $x(t) \neq 0$ for $0 < t < \tau$.

Of course, x is periodic with period 2τ and $(\alpha, \varphi) \in \Lambda$, $Q(\alpha, \varphi) = -\varphi$, $\tau(\alpha, \varphi) = \tau$ where $\varphi := x|_{[0,1]}$.

The converse of this statement is also true:

Lemma 1.5. *Let $f \in C(\mathbb{R})$ be odd and $(\alpha, \varphi) \in \Lambda$ with $Q(\alpha, \varphi) = -\varphi$. Then $x(\cdot, \alpha, \varphi)$ is a symmetric periodic solution of (αf) with period $2\tau(\alpha, \varphi)$.*

Proof. Since f is odd, we have $x(\cdot, \alpha, -\varphi) = -x(\cdot, \alpha, \varphi)$, $\tau(\alpha, -\varphi) = \tau(\alpha, \varphi)$, and $Q(\alpha, -\varphi) = -Q(\alpha, -\varphi)$. If $Q(\alpha, \varphi) = -\varphi$, $\tau := \tau(\alpha, \varphi)$, it follows that $x := x(\cdot, \alpha, \varphi)$ and $y := -x(\cdot + \tau, \alpha, \varphi)$ solve the same initial value problem (αf), hence by uniqueness $x = y$. By the definition of Λ we see, that x is a symmetric periodic solution in the sense of the previous definition. ∎

Now let $f \in C^1(\mathbb{R})$ and $Q(\alpha, \varphi) = -\varphi$ for some $(\alpha, \varphi) \in \Lambda$. Then φ is differentiable, and we can define the operators $W_\varphi^\alpha \in L(C, C)$ and $U_\varphi^\alpha \in L(C_0, C_0)$ by

$$W_\varphi^\alpha := DX(\tau(\alpha, \varphi), \alpha, \cdot)(\varphi), \qquad U_\varphi^\alpha := DQ(\alpha, \cdot)(\varphi).$$

If $\dot\varphi(0) \neq 0$, U_φ^α is the projection of W_φ^α from $C = C_0 \oplus \mathbb{R}\dot\varphi$ onto C_0:

Lemma 1.6. *$U_\varphi^\alpha \xi = W_\varphi^\alpha \xi - \dot\varphi\kappa$ for $\xi \in C_0$ where $\kappa := W_\varphi^\alpha \xi(0)/\dot\varphi(0) \in \mathbb{R}$.*

Proof. Differentiation of $Q(\alpha, \varphi) = X(\tau(\alpha, \varphi), \alpha, \varphi)$ yields

$$\begin{aligned} U_\varphi^\alpha &= \dot X(\tau(\alpha, \varphi), \alpha, \varphi) \cdot D\tau(\alpha, \cdot)(\varphi) + X_\varphi(\tau(\alpha, \varphi), \alpha, \varphi) \\ &= -\dot\varphi \cdot D\tau(\alpha, \cdot)(\varphi) + W_\varphi^\alpha. \end{aligned}$$

Let $\Gamma: C \ni \psi \mapsto \psi(0) \in \mathbb{R}$, $\Gamma \in L(C, \mathbb{R})$. Then $\Gamma(Q(\alpha, \varphi)) = 0$ since $Q(\alpha, \varphi) \in C_0$, and $0 = \Gamma \circ U^\alpha_\varphi = \Gamma \circ W^\alpha_\varphi - \Gamma(\dot\varphi) \cdot D\tau(\alpha, \cdot)(\varphi)$. This shows $D\tau(\alpha, \cdot)(\varphi) = (1/\dot\varphi(0))\, \Gamma \circ W^\alpha_\varphi$. ∎

If $x := x(\cdot, \alpha, \varphi)$, $w := \dot x$, $\tau := \tau(\alpha, \varphi)$, then $\dot x(t) = \alpha f(x(t-1))$ and Definition 1.2 gives $w = w(\cdot, \alpha, \varphi, \dot\varphi)$, and by Lemma 1.3 we have

$$W^\alpha_\varphi \dot\varphi = X_\varphi(\tau, \alpha, \varphi) = W(\tau, \alpha, \varphi) = w(\tau + \cdot)|_{[0,1]} = \dot x(\tau + \cdot)|_{[0,1]} = -\dot\varphi.$$

Thus -1 is an eigenvalue of W^α_φ with eigenfunction $-\dot\varphi$.

Kaplan and Yorke [9] proved in 1974, that a special class of symmetric periodic solutions can be derived from a Hamiltonian system of ordinary differential equations. Let us first define

DEFINITION 1.5. Let $f \in C(\mathbb{R})$ be odd and $\alpha \in \mathbb{R}$. We say that a continuous map $x: \mathbb{R}^+_0 \to \mathbb{R}$ is a *special symmetric solution* of (αf), if x is a symmetric periodic solution of (αf) with period 4 and if $x|_{[0,1]} \in C_0$ is strictly monotone.

If $x: \mathbb{R}^+_0 \to \mathbb{R}$ is a (special) symmetric solution, then it has a periodic continuation on $\mathbb{R}$, with $x(t+2) = -x(t)$ for all $t \in \mathbb{R}$. We call $z := x(1)$ the "amplitude" of x. Now let $y := x(\cdot - 1)$; then one easily sees

$$\begin{aligned} \dot x &= \alpha f(y), & x(0) &= 0, & x(1) &= z \\ \dot y &= -\alpha f(x), & y(0) &= -z, & y(1) &= 0. \end{aligned}$$

These equations are often called Kaplan–Yorke equations. The result of Kaplan and Yorke proves the converse:

THEOREM 1.1 (Kaplan and Yorke [9]). *Let $f \in C^k(\mathbb{R})$ be odd, $k \in \mathbb{N}$, $f(0) = 0$, $f'(0) < 0$ and $xf(x) < 0$ for $x \neq 0$. Then there are C^k-maps $(x, y): \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ and $\alpha: \mathbb{R} \to \mathbb{R}$, such that for all $(t, z) \in \mathbb{R} \times \mathbb{R}$ the following is true*:

(1) $\dot x(t, z) = \alpha(z) f(y(t, z))$, $x(0, z) = 0$, $x(1, z) = z$
$\dot y(t, z) = -\alpha(z) f(x(t, z))$, $y(0, z) = -z$, $y(1, z) = 0$

(2) $y(t, z) = x(t-1, z)$, $x(t+2, z) = -x(t, z)$, $x(-t, z) = -x(t, z)$, $y(-t, z) = y(t, z)$

(3) $F(x(t, z)) + F(y(t, z)) = F(z)$ *with* $F := \int_0^{\cdot} f(u)\, du$

(4) *if $z > 0$ then $x(\cdot, z)$, $y(\cdot, z)$ are strictly increasing on* $[0, 1]$.

Sketch of proof. Let $z > 0$ and $(X(\cdot, z), Y(\cdot, z)): \mathbb{R} \to \mathbb{R}^2$ be the solution of

(i) $\dot X = f(Y)$, $X(0) = 0$,
$\dot Y = -f(X)$, $Y(0) = -z$.

Then X, Y are C^k-maps and

(ii) $F(X)+F(Y)=F(z)$.

The main step, which we do not prove, is:
There is a unique $\alpha(z)>0$ with

(iii) $-z<Y(t,z)<0$ for $0<t<\alpha(z)$,

(iv) $Y(\alpha(z),z)=0$.

Because of (i, iii) $X(\cdot,z)$ is strictly increasing on $[0,\alpha(z)]$, and (ii, iv) gives $F(X(\alpha(z),z))=F(z)$. Since F is even and $F|_{\mathbb{R}_0^+}$ is strictly monotone, it follows that

(v) $0<X(t,z)<z$ for $0<t<\alpha(z)$,

(vi) $X(\alpha(z),z)=z$.

Now (i, iii, v) yield

(vii) $X(\cdot,z)$ and $Y(\cdot,z)$ are strictly increasing on $[0,\alpha(z)]$.

The implicit function theorem and $\dot{Y}(\alpha(z),z)=-f(z)\neq 0$ show that α is a C^k-map. Let $x(t,z):=X(\alpha(z)t,z)$ and $y(t,z):=Y(\alpha(z)t,z)$ for $z>0$, $t\in\mathbb{R}$. Then (1) follows from (i, iv, vi), (3) from (ii), and (4) from (vii). For $u:=-y(\cdot-1,z)$ and $v:=x(\cdot-1,z)$ one easily finds

$$\dot{u}=\alpha(z)\,f(v),\qquad u(1)=z=x(1,z)$$
$$\dot{v}=-\alpha(z)\,f(u),\qquad v(1)=0=y(1,z),$$

which gives $x(t,z)=u(t)=-y(t-1,z)$, $y(t,z)=v(t)=x(t-1,z)$. This proves (2); $x(-t,z)=-x(t,z)$ and $y(-t,z)=y(t,z)$ are shown by an analogous argument. ∎

COROLLARY 1.1. *For $z>0$, $x(\cdot,z)|_{\mathbb{R}_0^+}$ is a special symmetric solution of $(\alpha(z)f)$ with amplitude z.*

Hence let us define:

DEFINITION 1.6. Let $x(z):=x(\cdot,z)|_{[0,1]}\in C_0$ for $z\in\mathbb{R}$. Then we call $\mathrm{PB}(f):=\{(\alpha(z),x(z))\in\mathbb{R}^+\times C_0 \mid z\in\mathbb{R}^+\}\subseteq\mathbb{R}^+\times C_0$ the *primary branch* of symmetric period-4 solutions or simply the *Kaplan–Yorke branch* of f.

Here $\mathbb{R}\ni z\mapsto x(z)\in C_0$ is a C^k-map with derivative $x'(z)(t)=(\partial x/\partial z)(t,z)$ for $z\in\mathbb{R}$, $t\in[0,1]$. Hence the Kaplan–Yorke branch can be parameterized by the amplitude z in a differentiable way. Moreover $\mathrm{PB}(f)$ contains all special symmetric solutions. Let us state this result (without proof) in

LEMMA 1.7. *If* $x: \mathbb{R}_0^+ \to \mathbb{R}$ *is a special symmetric solution with amplitude* $z := x(1)$, *then* $x = x(\cdot, z)|_{\mathbb{R}_0^+}$.

In addition to the Theorem of Kaplan and Yorke one has $x(\cdot, 0) = y(\cdot, 0) = 0$, $\alpha(0) = -\pi/2f'(0)$ and $\alpha'(0) = 0$. Hence the Kaplan–Yorke branch bifurcates at $\alpha = -\pi/2f'(0)$ from the trivial solutions $x \equiv 0$ and has a vertical tangent there. Walther [15] proved that the direction of the branch, i.e., the sign of $\alpha''(0)$, is equal to the sign of $f'''(0)$. Meanwhile we found the sharper result

$$\alpha''(0) = \frac{\pi}{8} \frac{f'''(0)}{[f'(0)]^2}.$$

Thus $f'''(0) < 0$ implies that the bifurcation is backward. Moreover we were able to prove that for small amplitudes z, $x(\cdot, z)$ is stable (as a fixed point of the Poincaré map $P(\alpha(z), \cdot)$) if $f'''(0) > 0$ (bifurcation to the right), and unstable if $f'''(0) < 0$ (bifurcation to the left).

An asymptotic analysis of $\alpha(z)$ for $z \to \infty$ can be found in [4] and [13]. These results show, that for $f(x) = -x((1 + x^2)/(1 + x^4))$, $x \in \mathbb{R}$, $\alpha(z)$ has the qualitative behaviour of the diagram of Fig. 4.

This agrees with numerical studies of Hadeler [6] and has as a consequence that in general the Kaplan–Yorke branch cannot be parameterized by α in a unique way. Hence we have to use the amplitude z as bifurcation parameter instead of the delay parameter α.

Now we are searching for conditions for the existence of a bifurcation point on the Kaplan–Yorke branch where symmetric periodic solutions bifurcate. For this purpose we state an appropriate bifurcation equation, and apply the theorem on the bifurcation from a simple eigenvalue. From the preceding we know:

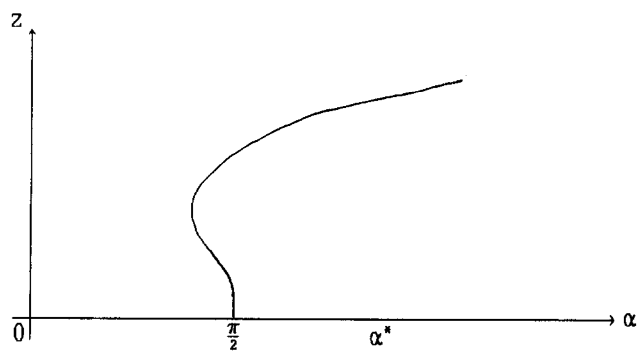

FIGURE 4

LEMMA 1.8. (1) *For* $z>0$ *we have* $(\alpha(z), x(z)) \in \Lambda$, $\tau(\alpha(z), x(z))=2$ *and* $Q(\alpha(z), x(z)) = -x(z)$.

(2) *Let* $h: \mathbb{R}^+ \times C_0 \ni (z, \varphi) \mapsto (\alpha(z), x(z)+\varphi) \in \mathbb{R}^+ \times C_0$; h *is continuously differentiable,* $h(\mathbb{R}^+ \times \{0\}) = PB(f) \subseteq \Lambda$ *and* $\Lambda^* := h^{-1}(\Lambda) \subseteq \mathbb{R}^+ \times C_0$ *is an open neighbourhood of* $\mathbb{R}^+ \times \{0\} \subseteq \mathbb{R}^+ \times C_0$.

(3) *Let* $g := (\Lambda^* \ni (z, \varphi) \mapsto Q(\alpha(z), x(z)+\varphi) + x(z) + \varphi \in C_0)$; g *is continuously differentiable,* $g(z, 0) = 0$ *for all* $z>0$, *and* $g(z, \varphi) = 0$ *for some* $(z, \varphi) \in \Lambda^*$ *implies that* $x(\cdot, \alpha(z), x(z)+\varphi)$ *is a symmetric periodic solution of* $(\alpha(z) f)$ *with period* $2\tau(\alpha(z), x(z)+\varphi)$ *and amplitude* $z+\varphi(1)$.

Hence we want to apply the theorem on the bifurcation from a simple eigenvalue to the bifurcation equation $g(\alpha, \varphi) = 0$. But the following problem arises: the map g does not have the differentiability conditions which are assumed, e.g., in [3]. For that reason we need a version of this theorem which fits to our situation.

THEOREM 1.2 (Bifurcation from a simple eigenvalue). *Let* $\mathbb{B}$ *be a Banach space,* $z^* \in \mathbb{R}$, $J \subseteq \mathbb{R}$ *an open interval of* z^* *and* $U \subseteq \mathbb{B}$ *an open neighbourhood of* $0 \in \mathbb{B}$. *Let* $G: J \times U \to \mathbb{B}$ *be a* C^1*-map with* $G|_{J \times \{0\}} = 0$. *Suppose that*

(1) *there is a continuous map*

$$H: W \to \mathbb{B},\ W := \{(s, z, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{B} \mid (z, sx) \in J \times U\}$$

with $DG(\cdot, sx)(z) = s \cdot H(s, z, x)$ *for all* $(s, z, x) \in W$.

(2) $A^* := DG(z^*, \cdot)(0) \in L(\mathbb{B}, \mathbb{B})$ *is a Fredholm operator with index* 0.

(3) *there is a* $b \in \mathbb{B}$ *with* $\ker A^* = \mathbb{R}b$ *and* $H(0, z^*, b) \notin \operatorname{im} A^*$.

Then z^* *is a differentiable bifurcation point of* G, *i.e., there is an open* $\hat{J} \times \hat{U} \subseteq \mathbb{R} \times \mathbb{B}$ *with* $(0, 0) \in \hat{J} \times \hat{U}$, *an open* $I \subseteq \mathbb{R}$ *with* $0 \in I$ *and a continuous map* $(z, x): I \to \hat{J} \times \hat{U}$ *such that*

(a) $z(0) = z^*$, $x(0) = 0$, $x(s) \neq 0$ *for* $s \in I \setminus \{0\}$,

(b) (z, x) *is continuously differentiable on* $I \setminus \{0\}$, x *is differentiable at* 0 *with* $\dot{x}(0) = b$.

(c) $G^{-1}(\{0\}) \cap (\hat{J} \times \hat{U}) = (\hat{J} \times \{0\}) \cup (z, x)(I)$.

Proof. (1) Since G is differentiable, Taylor's formula yields a continuous map $K: W \to \mathbb{B}$ with $G(z, sx) = s \cdot K(s, z, x)$ for $(s, z, x) \in W$, and we have $K(0, z, \cdot) = DG(z, \cdot)(0) \in L(\mathbb{B}, \mathbb{B})$ for $z \in J$. Let $E \subseteq \mathbb{B}$ be a closed subspace of $\mathbb{B}$ with $\mathbb{R}b \oplus E = \mathbb{B}$. We now follow the idea of Crandall and Rabinowitz [3] and apply the implicit function theorem to the map

$\bar{K}: \bar{W} \to \mathbb{B}$ where $\bar{W} := \{(s, z, e) \in \mathbb{R} \times \mathbb{R} \times E \mid (s, z, b+e) \in W\}$, $\bar{K}(s, z, e) := K(s, z, b+e)$ for $(s, z, e) \in \bar{W}$.

(2) To this end we first show: If $s \in \mathbb{R}$ and

$$W_s := \{(z, x) \in \mathbb{R} \times \mathbb{B} \mid (s, z, x) \in W\} \subseteq \mathbb{R} \times \mathbb{B},$$
$$K_s := (W_s \ni (z, x) \mapsto K(s, z, x) \in \mathbb{B}),$$

then K_s has partial derivatives with respect to z and x and

$$\frac{\partial}{\partial z} K_s(z, x) = H(s, z, x), \qquad \frac{\partial}{\partial x} K_s(z, x) = DG(z, \cdot)(sx).$$

This is obvious for $s \neq 0$ since $K_s(z, x) = (1/s)\, G(z, sx)$. Now let $s = 0$, then $W_0 = J \times \mathbb{B}$ and $K_0(z, x) = [DG(z, \cdot)(0)]x$ for $(z, x) \in W_0$. This shows $(\partial/\partial x)\, K_0(z, 0) = DG(z, \cdot)(0)$. If $u \in \mathbb{R}$ with $z + u \in J$ then

$$G(z+u, x) - G(z, x) = u \cdot \int_0^1 DG(\cdot, x)(z + tu)\, dt,$$

and since G is differentiable:

$$\begin{aligned} &K_0(z+u, x) - K_0(z, x) \\ &\quad = DG(z+u, \cdot)(0)x - DG(z, \cdot)(0)x \\ &\quad = \lim_{s \to 0} \frac{1}{s} [G(z+u, sx) - G(z, sx)] = \lim_{s \to 0} \frac{1}{s} \cdot u \cdot \int_0^1 DG(\cdot, sx)(z + tu)\, dt \\ &\quad = \lim_{s \to 0} u \cdot \int_0^1 H(s, z + tu, x)\, dt = u \cdot \int_0^1 H(0, z + tu, x)\, dt, \end{aligned}$$

hence,

$$\begin{aligned} \frac{\partial}{\partial z} K_0(z, x) &= \lim_{u \to 0} \frac{1}{u} [K(0, z+u, x) - K(0, z, x)] \\ &= \lim_{u \to 0} \int_0^1 H(0, z + tu, x)\, dt = H(0, z, x). \end{aligned}$$

(3) We have proved

(i) K_s is continuously differentiable and $W \ni (s, z, x) \mapsto DK_s(z, x) \in L(\mathbb{R} \times \mathbb{B}, \mathbb{B})$ is continuous since H and DG are continuous,

(ii) $DK_0(z^*, b)(\zeta, \xi) = \zeta \cdot H(0, z^*, b) + A^*\xi$ for all $(\zeta, \xi) \in \mathbb{R} \times \mathbb{B}$,

(iii) $H(0, z^*, b) = (\partial/\partial z)\, K_0(z^*, b) = d/dz\,|_{z^*} [DG(z, \cdot)(0)b]$, and analogously for $\bar{K}$:

(iv) $\bar{K}$ is continuous, $\bar{K}_s\colon \bar{W}_s \to \mathbb{B}$ with $\bar{W}_s := \{(z, e) \in \mathbb{R} \times \mathbb{B} \mid (s, z, e) \in \bar{W}\}$ is continuously differentiable for all $s \in \mathbb{R}$ and $\bar{W} \ni (s, z, e) \mapsto D\bar{K}_s(z, e) \in L(\mathbb{R} \times E, \mathbb{B})$ is continuous.

(v) Let $\bar{b} := H(0, z^*, b) \in \mathbb{B}$, then $D\bar{K}_0(z^*, 0)(\zeta, \xi) = \zeta \cdot \bar{b} + A^*\xi$ for $(\zeta, \xi) \in \mathbb{R} \times E$, and by our assumptions $D\bar{K}_0(z^*, 0) \in L(\mathbb{R} \times E, \mathbb{B})$ is an isomorphism.

(vi) $\bar{K}(0, z^*, 0) = K(0, z^*, b) = A^*b = 0$.

(4) Now the implicit function theorem (see, e.g., Theorem 4.2 in [17]) yields an open set $\bar{I} \times \bar{J} \times V \subseteq \bar{W} \subseteq \mathbb{R} \times \mathbb{R} \times E$ with $(0, z^*, 0) \in \bar{I} \times \bar{J} \times V$ and a continuous map $(z, e)\colon \bar{I} \to \bar{J} \times V$ with $z(0) = z^*$, $e(0) = 0$ and

$$\bar{K}^{-1}(\{0\}) \cap (\bar{I} \times \bar{J} \times V) = \operatorname{graph}(z, e).$$

Because of (v) we can assume that for $(s, z, e) \in \bar{I} \times \bar{J} \times V$ $D\bar{K}_s(z, e)$ is an isomorphism too. Since $\bar{K}$ is continuously differentiable on $(\bar{I} \backslash \{0\}) \times \bar{J} \times V$, we can apply the implicit function theorem to $\bar{K}$ at every point $(s, z(s), e(s))$, $s \in \bar{I} \backslash \{0\}$. This shows that (z, e) is a C^1-map on $\bar{I} \backslash \{0\}$.

(5) Define $x\colon \bar{I} \ni s \mapsto s \cdot (b + e(s)) \in \mathbb{B}$; x is continuous and of class C^1 on $\bar{I} \backslash \{0\}$. For $s = 0$ the derivative $\dot{x}(0) = b$ exists, but $\dot{x}$ is not necessarily continuous at $s = 0$. For $s \in \bar{I}$ we have

$$\begin{aligned} G(z(s), x(s)) &= G(z(s), s \cdot (b + e(s))) = s \cdot K(s, z(s), b + e(s)) \\ &= s \cdot \bar{K}(s, z(s), e(s)) = 0. \end{aligned}$$

Furthermore $x(s) \neq 0$ for $s \in \bar{I} \backslash \{0\}$, since otherwise $b = -e(s) \in \mathbb{R}b \cap E = \{0\}$ implies $b = 0$, and (3iii) gives the contradiction $H(0, z^*, 0) = 0 \in \operatorname{im} A^*$.

(6) Let $\varepsilon > 0$ such that $\{e \in E \mid \|e\| < \varepsilon\} \subseteq V$. Next we prove that there is an open set $\hat{J} \times \hat{U} \subseteq \mathbb{R} \times \mathbb{B}$ with $(z^*, 0) \in \hat{J} \times \hat{U}$ and $\hat{J} \subseteq \bar{J}$ such that for all $(w, y) \in G^{-1}(\{0\}) \cap (\hat{J} \times \hat{U})$ the following holds:

$$\text{If } y = tb + e \in \mathbb{R}b \oplus E, \text{ then } t \in \bar{I} \text{ and } \|e\| \leqslant \frac{\varepsilon}{2}\,|t|.$$

Proof. We only show: There is an open set $\hat{J} \times \hat{U} \subseteq \mathbb{R} \times \mathbb{B}$ such that $\|e\| \leqslant \varepsilon/2\,|t|$ for all $(w, t, e) \in \mathbb{R} \times \mathbb{R} \times E$ with $(w, tb + e) \in h^{-1}(\{0\}) \cap (\hat{J} \times \hat{U})$. Since the projection $\mathbb{R}b \oplus E \ni tb + e \mapsto \in \mathbb{R}$ is continuous, $\hat{J} \subseteq \bar{J}$ and $t \in \bar{I}$ can be achieved by making $\hat{J}$ and $\hat{U}$ smaller if necessary. Since $\mathbb{R} \times E \ni (\zeta, \eta) \mapsto \zeta\bar{b} + A^*\eta \in \mathbb{B}$ is an isomorphism, there is a $c > 0$ such that $\|\zeta\bar{b} + A^*\eta\| \geqslant c \cdot (|\zeta| + \|\eta\|)$ for all $(\zeta, \eta) \in \mathbb{R} \times E$. Choose $\varepsilon' \in\,]0, \frac{1}{3}c[$ with $2\varepsilon'\,\|b\|/(c - 3\varepsilon') \leqslant \frac{1}{2}\varepsilon$. Since DG is continuous at $(z^*, 0) \in \mathbb{R} \times \mathbb{B}$, there is an open set $\hat{J} \times \hat{U} \subseteq \mathbb{R} \times \mathbb{B}$ with $(z^*, 0) \in \hat{J} \times \hat{U}$ and $\|DG(w, \cdot)(y) - A^*\| < \varepsilon'$ for all $(w, y) \in \hat{J} \times \hat{U}$. In particular $\|DG(w, \cdot)(0) - A^*\| < \varepsilon'$ for all $w \in \hat{J}$, and for all (w_1, y_1), $(w_2, y_2) \in \hat{J} \times \hat{U}$ we have $\|DG(w_1, \cdot)(y_1) - DG(w_2, \cdot)(y_2)\| \leqslant \|DG(w_1, \cdot)(y_1) - A^*\| + \|DG(w_2, \cdot)(y_2) - A^*\| < 2\varepsilon'$. Since $DG(z^*, \cdot)(0)b = A^*b = 0$ and $\bar{b} = d/dz|_{z^*} [DG(z, \cdot)(0)b]$ we can assume $\|DG(w, \cdot)(0)b - (w - z^*) \cdot \bar{b}\| \leqslant c|w - z^*|$ for all $w \in \hat{J}$. Now let $(w, tb + e) \in G^{-1}(\{0\}) \cap (\hat{J} \times \hat{U})$ and $u := w - z^*$. Then

$$-[t \cdot u \cdot \bar{b} + A^*e] = [G(w, tb + e) - G(w, 0) - DG(w, \cdot)(0)(tb + e)] + [DG(w, \cdot)(0)(tb) - t \cdot u \cdot \bar{b}] + DG(w, \cdot)(0)e - A^*e,$$

and Taylor's formula gives

$$\begin{aligned} c \cdot (|t \cdot u| + \|e\|) &\leqslant \|-[t \cdot u \cdot \bar{b} + A^*e]\| \\ &\leqslant \|G(w, tb + e) - G(w, 0) - DG(w, \cdot)(0)(tb + e)\| \\ &\quad + |t| \cdot \|DG(w, \cdot)(0)b - u \cdot \bar{b}\| + \|DG(w, \cdot)(0) - A^*\| \, \|e\| \\ &\leqslant \left\| \int_0^1 [DG(w, \cdot)(\sigma tb + \sigma e) - DG(w, \cdot)(0)](tb + e) \, d\sigma \right\| \\ &\quad + |t| \cdot c \cdot |u| + \varepsilon' \|e\| \\ &\leqslant 2\varepsilon'(|t| \cdot \|b\| + \|e\|) + c|t \cdot u| + \varepsilon'\|e\| \\ &= 2\varepsilon'|t| \cdot \|b\| + 3\varepsilon'\|e\| + c|t \cdot u|. \end{aligned}$$

Thus we have $(c - 3\varepsilon') \cdot \|e\| \leqslant 2\varepsilon'|t| \cdot \|b\|$ resp. $\|e\| \leqslant 2\varepsilon'\|b\|/(c - 3\varepsilon')\, |t| \leqslant \frac{1}{2}\varepsilon\, |t|$.

(7) Let $I := (z, x)^{-1} (\hat{J} \times \hat{U})$. Then I is an open neighbourhood of $0 \in \mathbb{R}$ since $(z, x)(0) = (z^*, 0) \in \hat{J} \times \hat{U}$. It follows that

$$(\hat{J} \times \{0\}) \cup (z, x)(I) \subseteq G^{-1}(\{0\}) \cap (\hat{J} \times \hat{U}).$$

Now let $(w, y) \in G^{-1}(\{0\}) \cap (\hat{J} \times \hat{U})$ and $y = tb + e$ with $t \in \mathbb{R}$, $e \in E$. Then we have $t \in \bar{I}$ and $\|e\| \leqslant \varepsilon/2\, |t|$. If $t = 0$, then $\|e\| \leqslant 0$ resp. $e = 0$ and $y = 0$, hence $(w, y) \in \hat{J} \times \{0\}$. If $t \neq 0$, then $\|(1/t)e\| \leqslant \varepsilon/2 < \varepsilon$ resp. $(1/t)e \in V$ and $0 = G(w, y) = G(w, t \cdot (b + (1/t)e)) = t \cdot \bar{K}(t, w, (1/t)e)$, hence, $\bar{K}(t, w, (1/t)e) = 0$. Since $t \in \bar{I}$ and $w \in \hat{J} \subseteq \bar{J}$, $(1/t)e \in V$, it follows that $(t, w, (1/t)e) \in \bar{K}^{-1}(\{0\}) \cap (\bar{I} \times \bar{J} \times V) = \operatorname{graph}(z, e)$. Thus we have $w = z(t)$, $e = t \cdot e(t)$, $y = t \cdot (b + e(t)) = x(t)$, which shows that $(z(t), x(t)) = (w, y) \in \hat{J} \times \hat{U}$, $t \in I$ and $(w, y) = (z(t), x(t)) \in (z, x)(I)$. ∎

Remark 1.2. By part (3iii) of our proof, the map $J \ni z \mapsto DG(z, \cdot)(0)b \in \mathbb{B}$ is differentiable with $d/dz|_{z^*} [DG(z, \cdot)(0)b] = H(0, z^*, b)$.

Remark 1.3. We cannot prove that (z, x) is differentiable at $0 \in I$. Of course, the first assumption of the theorem is true, if G is of class C^2, or if the mixed partial derivative $(\partial/\partial x)(\partial/\partial z)G$ exists and is continuous (compare [3]).

In our situation we have $\mathbb{B} = C_0$, $G = g$, $A^* = Dg(z^*, \cdot)(0) = U(z^*) + 1_{C_0}$ with $U(z) := DQ(\alpha(z), \cdot)(x(z)) = U_{x(z)}^{\alpha(z)} \in L(C_0, C_0)$, and Theorem 1.2 can be restated as follows:

THEOREM 1.3. *Let $f \in C^2(\mathbb{R})$ be odd, $f(0) = 0$, $f'(0) < 0$, $xf(x) < 0$ for $x \neq 0$. Suppose there is a $z^* > 0$ such that:*

(1) $\alpha'(z^*) \neq 0$,

(2) $U^* := U(z^*)$ *has* -1 *as an eigenvalue and* $\dim \ker(U^* + 1_{C_0}) = 1$,

(3) *for some (and hence all)* $\xi \in \ker(U^* + 1_{C_0})$, $\xi \neq 0$:

$$\chi := U'(z^*)\xi \notin \operatorname{im}(U^* + 1_{C_0}),$$

(4) $U^* + 1_{C_0}$ *is a Fredholm operator with index* 0.

Then a differentiable branch of symmetric periodic solutions bifurcates at z^* from the primary branch of f, i.e., there is an open interval $I \subseteq \mathbb{R}$ with $0 \in I$, open neighbourhoods $J \subseteq \mathbb{R}$, $U \subseteq C_0$ of $\alpha^* := \alpha(z^*) \in \mathbb{R}$ and $x(z^*) \in C_0$ and a continuous map $(\bar{\alpha}, \bar{\varphi}) : I \to J \times U$ such that

(a) $\bar{\alpha}(0) = \alpha(z^*)$, $\bar{\varphi}(0) = x(z^*)$, *and $(\bar{\alpha}, \bar{\varphi})$ is continuously differentiable on $I \setminus \{0\}$,*

(b) $\bar{x}(\cdot, s) := x(\cdot, \bar{\alpha}(s), \bar{\varphi}(s)) : \mathbb{R}_0^+ \to \mathbb{R}$ *is a symmetric periodic solution of $(\bar{\alpha}(s) f)$ for all $s \in I$,*

(c) $(\bar{\alpha}(s), \bar{\varphi}(s)) \notin \mathrm{PB}(f)$ *for all* $s \in I \setminus \{0\}$,

(d) *if $x(\cdot, \beta, \varphi)$ is a symmetric periodic solution of (βf) for some $(\beta, \varphi) \in J \times U$, then either $\beta = \alpha(z)$, $\varphi = x(z)$ for some $z > 0$, or there is an $s \in I$ with $\beta = \bar{\alpha}(s)$, $\varphi = \bar{\varphi}(s)$.*

Proof. (1) Let $g : \Lambda^* \ni (z, \varphi) \mapsto Q(\alpha(z), x(z) + \varphi) + x(z) + \varphi \in C_0$ be as in Lemma 1.8(3). Then g is a C^1-map, $\mathbb{R}^+ \times \{0\} \subseteq \Lambda^*$ and $g|_{\mathbb{R}^+ \times \{0\}} = 0$. Define $\sigma : \Lambda^* \ni (z, \varphi) \mapsto \tau(\alpha(z), x(z) + \varphi) \in \mathbb{R}$; σ is a C^1-map, $\sigma(z, 0) = 2$ for all $z > 0$, and by the definition of Λ and τ we have

$$x(\sigma(z, \varphi), \alpha(z), x(z) + \varphi) = 0, \qquad \dot{x}(\sigma(z, \varphi), \alpha(z), x(z) + \varphi) \neq 0$$

for all $(z, \varphi) \in \Lambda^*$. Furthermore the definition of g and Q yields

$$g(z, \varphi) = x(z) + \varphi + X(\sigma(z, \varphi), \alpha(z), x(z) + \varphi),$$

for $(z, \varphi) \in \Lambda^*$, and differentiation with respect to z gives:

$$\begin{aligned}\frac{\partial g}{\partial z}(z, \varphi) = {} & x'(z) + \alpha'(z)\, X_\alpha(\sigma(z, \varphi), \alpha(z), x(z) + \varphi) \\ & + X_\varphi(\sigma(z, \varphi), \alpha(z), x(z) + \varphi)\, x'(z) \\ & + X_t(\sigma(z, \varphi), \alpha(z), x(z) + \varphi)\, \frac{\partial \sigma}{\partial z}(z, \varphi) \\ = {} & h_3(z, \varphi) + \bar{h}_1(z, \varphi)\, \frac{\partial \sigma}{\partial z}(z, \varphi),\end{aligned}$$

where $\bar{h}_1, h_3 \colon \Lambda^* \to C$ are defined by

$$\begin{aligned}\bar{h}_1(z, \varphi) &:= X_t(\sigma(z, \varphi), \alpha(z), x(z) + \varphi) = \dot{x}(\sigma(z, \varphi) + \cdot, \alpha(z), x(z) + \varphi)|_{[0,1]}, \\ h_3(z, \varphi) &:= x'(z) + \alpha'(z)\, X_\alpha(\sigma(z, \varphi), \alpha(z), x(z) + \varphi) \\ &\quad + X_\varphi(\sigma(z, \varphi), \alpha(z), x(z) + \varphi)\, x'(z), \qquad \text{for} \quad (z, \varphi) \in \Lambda^*.\end{aligned}$$

Since $\sigma > 1$, $\bar{h}_1$ is continuous, and because of Lemma 1.2 h_3 is continuously differentiable. Let $\Gamma := (C \ni \varphi \mapsto \varphi(0) \in \mathbb{R}) \in L(C, \mathbb{R})$. Then $\Gamma(g(z, \varphi)) = 0$ for all $(z, \varphi) \in \Lambda^*$ since $g(z, \varphi) \in C_0$, and differentiation with respect to z yields:

$$0 = \frac{\partial}{\partial z} \Gamma(g(z, \varphi)) = \Gamma\left(\frac{\partial g}{\partial z}(z, \varphi)\right) = \Gamma(h_3(z, \varphi)) + \Gamma(\bar{h}_1(z, \varphi)) \cdot \frac{\partial \sigma}{\partial z}(z, \varphi),$$

where

$$\begin{aligned}\Gamma(\bar{h}_1(z, \varphi)) &= \Gamma(\dot{x}(\sigma(z, \varphi) + \cdot, \alpha(z), x(z) + \varphi)|_{[0,1]}) \\ &= \dot{x}(\sigma(z, \varphi), \alpha(z), x(z) + \varphi) \neq 0.\end{aligned}$$

Define $h_2 := \Gamma \circ h_3 \colon \Lambda^* \to \mathbb{R}$, then we have: $\partial\sigma/\partial z = -(1/\Gamma \circ \bar{h}_1) h_2$, and thus

$$\frac{\partial g}{\partial z}(z, \varphi) = h_3 - \frac{1}{\Gamma \circ \bar{h}_1} \cdot \bar{h}_1 \cdot h_2 = h_3 + h_1 \cdot h_2$$

with

$$h_1 := -\frac{1}{\Gamma \circ \bar{h}_1} \cdot \bar{h}_1 \colon \Lambda^* \to C.$$

The maps h_3 and $h_2 = \Gamma \circ h_3$ are both continuously differentiable, h_1 is continuous and $h_2(z, 0) = -\Gamma(\bar{h}_1(z, 0))(\partial\sigma/\partial z)(z, 0) = 0$ since $\sigma(z, 0) = 2$ for all

$z \in \mathbb{R}^+$. Because of $0 = (\partial g/\partial z)(z, 0) = h_3(z, 0) + h_1(z, 0)\, h_2(z, 0)$ we also have $h_3(z, 0) = 0$ for all $z > 0$. Let $\Sigma := \{(s, z, \varphi) \in \mathbb{R} \times \mathbb{R} \times C \mid (z, s\varphi) \in \Lambda^*\}$, then Taylor's formula yields continuous maps $H_2 : \Sigma \to \mathbb{R}$, $H_3 : \Sigma \to C$ such that

$$h_2(z, s\varphi) = s \cdot H_2(s, z, \varphi), \qquad h_3(z, s\varphi) = s \cdot H_3(s, z, \varphi).$$

for all $(s, z, \varphi) \in \Sigma$. It follows that $(\partial g/\partial z)(z, s\varphi) = s \cdot H(s, z, \varphi)$ with the continuous map $H : \Sigma \ni (s, z, \varphi) \mapsto H_3(s, z, \varphi) + h_1(z, s\varphi) \cdot H_2(s, z\varphi) \in C_0$.

(2) Let $A^* := Dg(z^*, \cdot)(0) = DQ(\alpha(z^*), \cdot)(x(z^*)) + 1_{C_0} = U^* + 1_{C_0}$; then A^* is a Fredholm operator with index 0. Since $\dim \ker(U^* + 1_{C_0}) = 1$, there is a $\xi \in C_0$ with $\ker A^* = \mathbb{R}\xi$, $\xi \neq 0$, and $d/dz|_{z^*} [Dg(z, \cdot)(0)\xi] = U'(z^*)\xi = \chi \notin \operatorname{im} A^*$, (cmp. Remark 1.2). Now we can apply Theorem 1.2 and find open neighbourhoods $\hat{J} \subseteq \mathbb{R}^+$ of z^*, $\hat{U} \subseteq C_0$ of $0 \in C_0$, $\hat{I} \subseteq \mathbb{R}$ of $0 \in \mathbb{R}$ and a continuous map $(\bar{z}, \bar{\psi}) : \hat{I} \to \hat{J} \times \hat{U}$ such that:

(i) $\bar{z}(0) = z^*$, $\bar{\psi}(0) = 0$, $(\bar{z}, \bar{\psi})$ is continuously differentiable on $\hat{I} \backslash \{0\}$, $\bar{\psi}$ is differentiable at $0 \in \hat{I}$ with $\bar{\psi}'(0) = \xi \neq 0$, $\bar{\psi}'(0) \in \ker A^*$,

(ii) $(\bar{z}(s), \bar{\psi}(s)) \in \Lambda^*$ and $g(\bar{z}(s), \bar{\psi}(s)) = 0$ for all $s \in \hat{I}$,

(iii) $\bar{\psi}(s) \neq 0$ for $s \in \hat{I} \backslash \{0\}$,

(iv) $g^{-1}(\{0\}) \cap (\hat{J} \times \hat{U}) = (\hat{J} \times \{0\}) \cup (\bar{z}, \bar{\psi})(\hat{I})$.

Now define

$$\hat{\alpha} := (\hat{I} \ni s \mapsto \alpha(\bar{z}(s)) \in \mathbb{R}), \qquad \hat{\varphi} := (\hat{I} \ni s \mapsto x(\bar{z}(s)) + \bar{\psi}(s) \in C_0);$$

$\hat{\alpha}$ and $\hat{\varphi}$ are continuous and continuously differentiable on $\hat{I} \backslash \{0\}$, $\hat{\alpha}(0) = \alpha^*$, and $\hat{\varphi}(0) = x(z^*)$. Next we construct I, J, and U.

Since $\alpha'(z^*) \neq 0$, there are open intervals $J' \subseteq \hat{J}$, $J'' \subseteq \mathbb{R}$ with $z^* \in J'$, $\alpha^* \in J''$, such that $J' \ni z \mapsto \alpha(z) \in J''$ is homeomorphic. This remains true for the map

$$H : J' \times C_0 \ni (z, \varphi) \mapsto (\alpha(z), x(z) + \varphi) \in J'' \times C_0.$$

Since $(\alpha^*, x(z^*)) = H(z^*, 0) \in H(J' \times \hat{U})$, there are open sets $J \subseteq J''$, $U \subseteq C_0$ with $(\alpha^*, x(z^*)) \in J \times U \subseteq H(J' \times \hat{U})$. Because $\Xi : C_0 \ni \varphi \mapsto \varphi(1) \in \mathbb{R}$ is linear and continuous with $\Xi(x(z^*)) = z^* \in J'$, $\Xi^{-1}(J') \subseteq C_0$ is an open neighbourhood of $x(z^*)$. Since $x(z^*) \in U$ we can therefore assume $U \subseteq \Xi^{-1}(J')$, i.e., $\varphi(1) \in J'$ for all $\varphi \in U$. Furthermore $J''' := \alpha^{-1}(J) \cap J' \subseteq J'$ is an open neighbourhood of z^* since $\alpha^* \in J$. Now define $I := (\bar{z}, \hat{\varphi})^{-1}(J''' \times U) \subseteq \hat{I}$; since $(\bar{z}, \hat{\varphi}) : \hat{I} \to \hat{J} \times C_0$ is continuous, I is open, and from $(\bar{z}(0), \hat{\varphi}(0)) = (z^*, x(z^*)) \in J''' \times U$ we derive $0 \in I$. Let $s \in I$, i.e., $(\bar{z}(s), \hat{\varphi}(s)) \in J''' \times U$ or $\hat{\varphi}(s) \in U$ and $\bar{z}(s) \in \alpha^{-1}(J)$, resp. $\hat{\alpha}(s) = \alpha(\bar{z}(s)) \in J$. It follows that

$$(\bar{\alpha}, \bar{\varphi}) := (I \ni s \mapsto (\hat{\alpha}(s), \hat{\varphi}(s)) = (\alpha(\bar{z}(s)), x(\bar{z}(s)) + \bar{\psi}(s)) \in J \times U)$$

is defined, continuous, and continuously differentiable on $I\backslash\{0\}$. Moreover

(a) $\bar{\alpha}(0)=\hat{\alpha}(0)=\alpha^*$, $\bar{\varphi}(0)=\hat{\varphi}(0)=x(z^*)$.

(b) From (ii) we find for $s\in I\subseteq \hat{I}$:

$$0=g(\bar{z}(s),\bar{\psi}(s))=Q(\alpha(\bar{z}(s)),x(\bar{z}(s))+\bar{\psi}(s))+x(\bar{z}(s))+\bar{\psi}(s)$$
$$=Q(\bar{\alpha}(s),\bar{\varphi}(s))+\bar{\varphi}(s),$$

and hence $Q(\bar{\alpha}(s),\bar{\varphi}(s))=-\bar{\varphi}(s)$. Thus by Lemma 1.5 $\bar{x}(\cdot,s)=x(\cdot,\bar{\alpha}(s),\bar{\varphi}(s))$ is a symmetric periodic solution of $(\bar{\alpha}(s)f)$.

(c) Suppose $(\bar{\alpha}(s),\bar{\varphi}(s))\in \mathrm{PB}(f)$ for some $s\in I$. We prove $s=0$. First of all there is a $z>0$ with $\bar{\alpha}(s)=\alpha(z)$ and $\bar{\varphi}(s)=x(z)$. Since $(\bar{\alpha}(s),\bar{\varphi}(s))\in J\times U$ we have $\bar{\alpha}(s)=\alpha(\bar{z}(s))\in J$ and $\bar{\varphi}(s)\in U$, and thus $z=x(1,z)=\bar{\varphi}(s)(1)\in J'$. Now $s\in I$ implies $\bar{z}(s)\in J'''\subseteq J'$, hence we have $\bar{z}(s)$, $z\in J'$. Since α is injective on J' and $\alpha(\bar{z}(s))=\alpha(z)$, it follows that $\bar{z}(s)=z$ and thus $x(z)=\bar{\varphi}(s)=x(\bar{z}(s))+\bar{\psi}(s)=x(z)+\bar{\psi}(s)$. This shows $\bar{\psi}(s)=0$, and (iii) gives $s=0$.

(d) Suppose $x(\cdot,\beta,\varphi)$ is a symmetric periodic solution for some $(\beta,\varphi)\in J\times U$. Since $J\times U\subseteq H(J'\times\hat{U})$, there is a $z\in J'\subseteq\hat{J}$ and a $\psi\in\hat{U}$ with $\beta=\alpha(z)$, $\varphi=x(z)+\psi$. Furthermore $g(z,\psi)=Q(\alpha(z),x(z)+\psi)+x(z)+\psi=Q(\beta,\varphi)+\varphi=0$, and (iv) gives $(z,\psi)\in g^{-1}(\{0\})\cap(\hat{J}\times\hat{U})=(\hat{J}\times\{0\})\cup(\bar{z},\bar{\psi})(\hat{I})$. Hence either $(z,\psi)\in\hat{J}\times\{0\}$ or $(z,\psi)\in(\bar{z},\bar{\psi})(\hat{I})$.
If $(z,\psi)\in\hat{J}\times\{0\}$, then $\psi=0$, and hence $\beta=\alpha(z)$, $\varphi=x(z)$; if $(z,\psi)\in(\bar{z},\bar{\psi})(\hat{I})$, then there is a $s\in\hat{I}$ with $z=\bar{z}(s)$, $\psi=\bar{\psi}(s)$, and we have $\hat{\varphi}(s)=x(\bar{z}(s))+\bar{\psi}(s)=x(z)+\psi=\varphi\in U$, $\bar{z}(s)=z\in J'$, $\alpha(\bar{z}(s))=\alpha(z)=\beta\in J$. It follows that $\bar{z}(s)\in\alpha^{-1}(J)\cap J'=J'''$ and thus $(\bar{z}(s),\hat{\varphi}(s))\in J'''\times U$. This proves $s\in(\bar{z},\hat{\varphi})^{-1}(J'''\times U)=I$ and $\beta=\alpha(\bar{z}(s))=\bar{\alpha}(s)$, $\varphi=\hat{\varphi}(s)=\bar{\varphi}(s)$. ∎

Remark 1.4. Since U^* is compact (compare, e.g., [7]), $U^*+1_{C_0}$ is always a Fredholm operator with index 0. We shall obtain this independently in Corollary 3.1.

In the following we shall study the conditions of Theorem 1.3 and try to simplify them.

2. The Eigenvalue -1

In this section we always assume that

$$f\in C^2(\mathbb{R})\text{ is odd, } f(0)=0,\ f'(0)<0, \text{ and } xf(x)<0 \text{ for } x\neq 0.$$

Furthermore, let $x(\cdot,z)$, $y(\cdot,z)$ and $\alpha(z)$ be as in the Theorem of Kaplan and Yorke. Then

$$x(\cdot,z)|_{\mathbb{R}_0^+}=x(\cdot,\alpha(z),x(z)).$$

In addition to $U(z) := U^{\alpha(z)}_{x(z)} = DQ(\alpha(z), \cdot)(x(z)) \in L(C_0, C_0)$ we need the following operators:

$$V(z) := DX(1, \alpha(z), \cdot)(x(z)) = X_\varphi(1, \alpha(z), x(z)) = W(1, \alpha(z), x(z)) \in L(C, C),$$

$$W(z) := W^{\alpha(z)}_{x(z)} = DX(2, \alpha(z), \cdot)(x(z)) = X_\varphi(2, \alpha(z), x(z))$$
$$= W(2, \alpha(z), x(z)) \in L(C, C).$$

We have to find conditions for the eigenvalue -1 of $U(z)$. Walther [16] was the first who studied this kind of problem. His starting point is to describe the spectrum of the operator $U(z)$ with the help of a boundary value problem for a (two-dimensional) linear system of ordinary differential equations. It is not hard to find this system, but Walther was the first who drew conclusions by comparing the degrees of the Poincaré maps $P(\alpha, \cdot)$ for α near $-\pi/2f'(0)$ and for $\alpha \to \infty$. Thus he was able to prove the existence of a bifurcation point on $\mathrm{PB}(f)$ where *non*-symmetric periodic solutions do appear (Theorem 6.1 in [16]). We also study the boundary value problem mentioned above, but we are only interested in the eigenvalue -1. By Theorem 1.3 we have to describe not only $\ker(U(z) + 1_{C_0})$ and $\mathrm{im}(U(z) + 1_{C_0})$, but also the derivative of $U(z)$. Thus our situation is more complicated than by Walther since we need more information on $U(z)$; a substantial help is the fact, that -1 is always an eigenvalue of $W(z)$. The definition of $V(z)$, $W(z)$, Definition 1.2, Lemma 1.3, and the symmetry properties of $x(\cdot, z)$ immediately yield

LEMMA 2.1. *Let $z > 0$ and $\alpha := \alpha(z)$. Then for $\xi \in C$ the following hold:*

(1) $V(z)\xi = \xi(1) + \int_0^{\cdot} \alpha f'(x(\cdot, z))\xi \, ds \in C$, $\cdot \in [0, 1]$,

(2) $W(z)\xi = \eta(1) + \int_0^{\cdot} \alpha f'(y(\cdot, z))\eta \, ds \in C$, $\cdot \in [0, 1]$ *where* $\eta := V(z)\xi$,

(3) $U(z)\xi = W(z)\xi - \dot{x}(z)(1/\dot{x}(0, z))[W(z)\xi](0) \in C_0$ *for* $\xi \in C_0$ *where* $\dot{x}(z) := \dot{x}(\cdot, z)|_{[0,1]}$ *and* $\dot{x}(0, z) = -\alpha f(z) \neq 0$.

In particular $V(z)\xi$, $W(z)\xi$: $[0, 1] \to \mathbb{R}$ are continuously differentiable.

The question when -1 is an eigenvalue of $W(z)$ can be answered as follows:

LEMMA 2.2. *Let $\xi \in C$. Then $W(z)\xi = -\xi$ if and only if ξ is differentiable and if there is a differentiable $\eta \in C$ such that*

$$\begin{aligned} \dot{\eta} &= \alpha f'(x(\cdot, z))\xi, \qquad & \eta(0) &= \xi(1) \\ \dot{\xi} &= -\alpha f'(y(\cdot, z))\eta, \qquad & \xi(0) &= -\eta(1). \end{aligned} \tag{*}$$

In this case we have $\eta = V(z)\xi$.

This follows immediately from Lemma 2.1(1, 2).

COROLLARY 2.1. $W(z)\,\dot{x}(z) = -\dot{x}(z)$.

Proof. Define $\xi := \dot{x}(z)$ and $\eta := -\dot{y}(z)$ $(= -\dot{y}(\cdot, z)|_{[0,1]})$. Then differentiation of $\dot{x} = \alpha f(y)$, $\dot{y} = -\alpha f(x)$ yields the assumption. ∎

We have reduced the eigenvalue problem of $W(z)$ for the eigenvalue -1 to the linear boundary value problem (*). Next we do the same for $U(z)$:

LEMMA 2.3. *Let $z > 0$ and $\alpha := \alpha(z)$.*

(1) *If $\xi \in C_0$ and $U(z)\xi = -\xi$, then for $\eta := V(z)\xi$, $\mu := \eta(1)/\dot{x}(0, z)$ the following holds*:

$$\begin{aligned} \dot{\eta} &= \alpha f'(x(\cdot, z))\xi \\ \dot{\xi} &= -\alpha f'(y(\cdot, z))\eta + \mu\ddot{x}(\cdot, z) \\ \eta(0) = \xi(1), &\quad \xi(0) = 0, \quad \eta(1) = \mu\dot{x}(0, z). \end{aligned} \tag{**}$$

(2) *Suppose that $\xi \in C$ is differentiable and there is a $\mu \in \mathbb{R}$ and a differentiable $\eta \in C$ which fulfil (**). Then $\xi \in C_0$ and $U(z)\xi = -\xi$.*

Proof. (1) Let $\xi \in C_0$ and $U(z)\xi = -\xi$. By Lemma 2.1(1) we have $\dot{\eta} = \alpha f'(x)\xi$, and $\eta(0) = \xi(1)$ for $\eta := V(z)\xi$. From $\xi \in C_0$ we see $\xi(0) = 0$, and $\mu := \eta(1)/\dot{x}(0, z)$ gives $\eta(1) = \mu\dot{x}(0, z)$. Now by Lemma 2.1(2, 3) we find

$$\begin{aligned} \xi = -U(z)\xi &= -W(z)\xi + \dot{x}\frac{1}{\dot{x}(0, z)}[W(z)\xi](0) \\ &= -W(z)\xi + \dot{x}\frac{\eta(1)}{\dot{x}(0, z)} = -W(z)\xi + \mu\dot{x}, \end{aligned}$$

and differentiation yields: $\dot{\xi} = -\alpha f'(y)\eta + \mu\ddot{x}$.

(2) Let $\xi \in C$ be differentiable and suppose there is a differentiable $\eta \in C$ and a $\mu \in \mathbb{R}$ such that (**) is fulfilled. Then $\xi(0) = 0$ implies $\xi \in C_0$. Since $\dot{\eta} = \alpha f'(x)\xi$ and $\eta(0) = \xi(1)$, Lemma 2.1(1) shows $\eta = V(z)\xi$. From $\mu = \eta(1)/\dot{x}(0, z)$, $W(z)\,\xi(0) = \eta(1)$ and Lemma 2.1(3) we derive: $U(z)\xi = W(z)\xi - \mu\dot{x}$. Now differentiation and Lemma 2.1(2) yield $(U(z)\xi)^{\cdot} = \alpha f'(y)\eta - \mu\ddot{x} = -\dot{\xi}$, i.e., $U(z)\xi + \xi$ is constant. But $U(z)\,\xi(0) + \xi(0) = 0$ since ξ, $U(z)\xi \in C_0$ and hence $U(z)\xi + \xi = 0$. ∎

Before we study the system (**) let us define the fundamental solution of (*):

DEFINITION 2.1. For $z > 0$ and $\alpha := \alpha(z)$ let

$$A(\cdot, z) := \begin{pmatrix} 0 & \alpha f'(x(\cdot, z)) \\ -\alpha f'(y(\cdot, z)) & 0 \end{pmatrix} : \mathbb{R} \to L(\mathbb{R}^2, \mathbb{R}^2)$$

and let

$$S(\cdot, z) := \begin{pmatrix} p(\cdot, z) & u(\cdot, z) \\ q(\cdot, z) & v(\cdot, z) \end{pmatrix} : \mathbb{R} \to L(\mathbb{R}^2, \mathbb{R}^2)$$

be the (fundamental) solution of $\dot{S} = A(\cdot, z)S$, $S(0) = 1_{\mathbb{R}^2}$.

Remark 2.1. In components we have

$$\dot{p} = \alpha f'(x) q, \quad p(0, z) = 1, \qquad \dot{u} = \alpha f'(x) v, \quad u(0, z) = 0$$
$$\dot{q} = -\alpha f'(y) p, \quad q(0, z) = 0, \qquad \dot{v} = -\alpha f'(y) u, \quad v(0, z) = 1.$$

The study of the eigenvalue -1 is considerably simplified by the facts, that we know explicitly the right-hand column of $S(\cdot, z)$, and that the left-hand column possesses a certain symmetry property. This is important for our further calculations.

LEMMA 2.4. *Let $z > 0$ and $\alpha := \alpha(z)$. Then*

(1) $$u(\cdot, z) = \frac{1}{\alpha f(z)} \dot{y}(\cdot, z) = -\frac{f(x(\cdot, z))}{f(z)},$$

$$v(\cdot, z) = -\frac{1}{\alpha f(z)} \dot{x}(\cdot, z) = -\frac{f(y(\cdot, z))}{f(z)}.$$

(2) $$p(t, z) = -\frac{\partial y}{\partial z}(t, z) + \frac{\alpha'(z)}{\alpha(z)} t\dot{y}(t, z)$$

$$q(t, z) = \frac{\partial x}{\partial z}(t, z) - \frac{\alpha'(z)}{\alpha(z)} t\dot{x}(t, z)$$

for all $t \in \mathbb{R}$.

(3) $$p(1-t, z) = q(t, z) + \frac{\alpha'(z)}{\alpha(z)} \dot{x}(t, z)$$

$$q(1-t, z) = p(t, z) - \frac{\alpha'(z)}{\alpha(z)} \dot{y}(t, z)$$

for all $t \in \mathbb{R}$.

(4) $$S(1, z) = \begin{pmatrix} p(1, z) & u(1, z) \\ q(1, z) & v(1, z) \end{pmatrix} = \begin{pmatrix} -\alpha'(z) f(z) & -1 \\ 1 & 0 \end{pmatrix}.$$

(5) $\det S(t, z) = p(t, z)\, v(t, z) - q(t, z)\, u(t, z) = 1$ *for all $t \in \mathbb{R}$ and*

$$S(t, z)^{-1} = \begin{pmatrix} v(t, z) & -u(t, z) \\ -q(t, z) & p(t, z) \end{pmatrix}.$$

Proof. (1) follows by differentiation of $\dot{x}=\alpha f(y)$, $\dot{y}=-\alpha f(x)$ since $\dot{x}(0,z)=-\alpha f(z)$ and $\dot{y}(0,z)=0$.

(2) Define

$$\bar{p}(t):=-\frac{\partial y}{\partial z}(t,z)+\frac{\alpha'(z)}{\alpha(z)}t\dot{y}(t,z) \qquad \text{and} \qquad \bar{q}(t):=\frac{\partial x}{\partial z}(t,z)-\frac{\alpha'(z)}{\alpha(z)}t\dot{x}(t,z).$$

Since $x(0,z)=0$, $y(0,z)=-z$ we have

$$\bar{p}(0)=-\frac{\partial y}{\partial z}(0,z)+0=1 \qquad \text{and} \qquad \bar{q}(0)=\frac{\partial x}{\partial z}(0,z)-0=0,$$

and $\dot{y}=-\alpha f(x)$ yields (we omit the arguments):

$$\begin{aligned}
\dot{\bar{p}}-\alpha f'(x)\bar{q} &= -\frac{\partial\dot{y}}{\partial z}+\frac{\alpha'(z)}{\alpha(z)}\dot{y}+\frac{\alpha'(z)}{\alpha(z)}t\ddot{y}-\alpha f'(x)\frac{\partial x}{\partial z}+\alpha f'(x)\frac{\alpha'(z)}{\alpha(z)}t\dot{x}\\
&= -\frac{\partial\dot{y}}{\partial z}-\alpha'(z)f(x)+\frac{\alpha'(z)}{\alpha(z)}t[\ddot{y}+\alpha f'(x)\dot{x}]-\alpha\frac{\partial}{\partial z}[f(x)]\\
&= -\frac{\partial}{\partial z}[\dot{y}+\alpha f(x)]+\frac{\alpha'(z)}{\alpha(z)}t[\dot{y}+\alpha f(x)]^{\cdot}\\
&=0.
\end{aligned}$$

Analogously one finds $\dot{\bar{q}}+\alpha f'(y)\bar{p}=0$, and hence $\bar{p}=p(\cdot,z)$, $\bar{q}=q(\cdot,z)$.

(3) From Theorem 1.1(2) we see $x(1-t,z)=-x(t-1,z)=-y(t,z)$ for $t\in\mathbb{R}$. Thus

$$\frac{\partial x}{\partial z}(1-t,z)=-\frac{\partial y}{\partial z}(t,z), \qquad \dot{x}(1-t,z)=\dot{y}(t,z), \qquad \text{and}$$

$$\begin{aligned}
q(1-t,z) &= \frac{\partial x}{\partial z}(1-t,z)-\frac{\alpha'(z)}{\alpha(z)}(1-t)\,\dot{x}(1-t,z)\\
&= -\frac{\partial y}{\partial z}(t,z)-\frac{\alpha'(z)}{\alpha(z)}(1-t)\,\dot{y}(t,z)\\
&= p(t,z)-\frac{\alpha'(z)}{\alpha(z)}\,\dot{y}(t,z);
\end{aligned}$$

$$p(1-t,z)=q(1-(1-t),z)+\frac{\alpha'(z)}{\alpha(z)}\,\dot{y}(1-t,z)=q(t,z)+\frac{\alpha'(z)}{\alpha(z)}\,\dot{x}(t,z).$$

(4) From (3) we find

$$p(1,z)=q(0,z)+\frac{\alpha'(z)}{\alpha(z)}\,\dot{x}(0,z)=0+\frac{\alpha'(z)}{\alpha(z)}[-\alpha(z)\,f(z)]=-\alpha'(z)\,f(z),$$

and $q(1, z) = p(0, z) - (\alpha'(z)/\alpha(z))\, \dot{y}(0, z) = 1 - 0 = 1$. Now (1) shows $u(1, z) = -1$ and $v(1, z) = 0$.

(5) Immediate from $\operatorname{Sp} A(\cdot, z) = 0$ and $\det S(0, z) = 1$. ∎

Next we solve the initial value problem which belongs to the inhomogenous system (**) of Lemma 2.3:

LEMMA 2.5. *Let $z > 0$, $\alpha = \alpha(z)$, $(\eta_0, \xi_0, \mu) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, and let (η, ξ) be the solution of*

$$\dot{\eta} = \alpha f'(x(\cdot, z))\xi, \qquad \eta(0) = \eta_0$$
$$\dot{\xi} = -\alpha f'(y(\cdot, z))\eta + \mu \ddot{x}(\cdot, z), \qquad \xi(0) = \xi_0.$$

Then

(1)
$$\eta = p(\cdot, z) \cdot \left[\eta_0 - \mu \int_0^\cdot u(s, z)\, \ddot{x}(s, z)\, ds \right] + u(\cdot, z) \cdot \left[\xi_0 + \mu \int_0^\cdot p(s, z)\, \ddot{x}(s, z)\, ds \right]$$
$$\xi = q(\cdot, z) \cdot \left[\eta_0 - \mu \int_0^\cdot u(s, z)\, \ddot{x}(s, z)\, ds \right] + v(\cdot, z) \cdot \left[\xi_0 + \mu \int_0^\cdot p(s, z)\, \ddot{x}(s, z)\, ds \right]$$

(2)
$$\eta(1) = p(1, z)\eta_0 - \xi_0 - \mu \cdot \left[p(1, z) \int_0^1 u(s, z)\, \ddot{x}(s, z)\, ds + \int_0^1 p(s, z)\, \ddot{x}(s, z)\, ds \right]$$
$$\xi(1) = \eta_0 - \mu \int_0^1 u(s, z)\, \ddot{x}(s, z)\, ds.$$

Proof. (1) Let

$$a := \eta_0 - \mu \int_0^\cdot u(s, z)\, \ddot{x}(s, z)\, ds, \qquad a(0) = \eta_0,$$
$$b := \xi_0 + \mu \int_0^\cdot p(s, z)\, \ddot{x}(s, z)\, ds, \qquad b(0) = \xi_0.$$

and

$$\bar{\eta} := p(\cdot, z) \cdot a + u(\cdot, z) \cdot b \qquad \text{and} \qquad \bar{\xi} := q(\cdot, z) \cdot a + v(\cdot, z) \cdot b.$$

Then $\bar{\eta}(0)=1\cdot\eta_0+0\cdot\xi_0=\eta_0$, $\bar{\xi}(0)=0\cdot\eta_0+1\cdot\xi_0=\xi_0$, and a short calculation which uses Lemma 2.4(5) shows $\dot{\bar{\eta}}=\alpha f'(x)\bar{\xi}$, $\dot{\bar{\xi}}=-\alpha f'(y)\bar{\eta}+\mu\ddot{x}$, from which $\bar{\eta}=\eta$, $\bar{\xi}=\xi$ follows.

(2) From (1) with Lemma 2.4(4). ∎

In addition to this we calculate the integral $\int_0^1 p(s,z)\,\ddot{x}(s,z)\,ds$ which appears in Lemma 2.5(2):

LEMMA 2.6. *Let $z>0$ and $\alpha'(z)\cdot\int_0^1 u(s,z)\,\ddot{x}(s,z)\,ds=0$. Then*

$$\int_0^1 p(s,z)\,\ddot{x}(s,z)\,ds=-\frac{1}{2}\,\dot{x}(0,z)=\frac{1}{2}\,\alpha(z)\,f(z)\neq 0.$$

Proof. First of all $p(s,z)\,\ddot{x}(s,z)+\dot{q}(s,z)\,\dot{y}(s,z)=0$ for $s\in\mathbb{R}$ since $\dot{q}=-\alpha f'(y)\,p$ and $\dot{x}=\alpha f(y)$ yield

$$p\ddot{x}+\dot{q}\dot{y}=p\ddot{x}-\alpha f'(y)\,\dot{y}p=p\cdot[\ddot{x}-\alpha f'(y)\,\dot{y}]=p\cdot[\dot{x}-\alpha f(y)]^{\cdot}=0.$$

Because of $\alpha'(z)\int_0^1 u(s,z)\,\ddot{x}(s,z)\,ds=0$ and $p(1,z)=-\alpha'(z)\,f(z)$ it follows that $p(1,z)\cdot\int_0^1 u(s,z)\,\ddot{x}(s,z)\,ds=0$. Define

$$\begin{aligned} v(t):=&\left(\int_0^{1-t}+\int_t^1\right)\ddot{x}(s,z)\,p(s,z)\,ds\\ &+p(1,z)\int_t^1 u(s,z)\,\ddot{x}(s,z)\,ds+q(1-t,z)\,\dot{y}(1-t,z).\end{aligned}$$

Then $v(1)=0$ since $q(0,z)=0$. Theorem 1.1 shows

$$\begin{aligned} y(t,z)=x(t-1,z)&\Rightarrow \dot{y}(t,z)=\dot{x}(t-1,z),\ \ddot{y}(t,z)=\ddot{x}(t-1,z)\\ &\Rightarrow \ddot{y}(1-t,z)=\ddot{x}(1-t-1,z)=\ddot{x}(-t,z)=-\ddot{x}(t,z),\end{aligned}$$

for $t\in\mathbb{R}$, and $q(1-t,z)=p(t,z)+p(1,z)\,u(t,z)$ from Lemma 2.4(3) gives

$$\begin{aligned} \dot{v}(t)=&-\ddot{x}(1-t,z)\,p(1-t,z)-\ddot{x}(t,z)\,p(t,z)-p(1,z)\,u(t,z)\,\ddot{x}(t,z)\\ &-\dot{q}(1-t,z)\,\dot{y}(1-t,z)-q(1-t,z)\,\ddot{y}(1-t,z)\\ =&-\ddot{x}(1-t,z)\,p(1-t,z)-[p(t,z)+p(1,z)\,u(t,z)]\,\ddot{x}(t,z)\\ &-\dot{q}(1-t,z)\,\dot{y}(1-t,z)+q(1-t,z)\,\ddot{x}(t,z)\\ =&\,\ddot{x}(t,z)[q(1-t,z)-p(1,z)\,u(t,z)-p(t,z)]\\ &-[\ddot{x}(\cdot,z)\,p(\cdot,z)+\dot{q}(\cdot,z)\,\dot{y}(\cdot,z)](1-t)\\ =&\,0,\end{aligned}$$

since both brackets vanish. Hence $v(t)=0$ for all $t\in\mathbb{R}$. For $t=0$ we find

$$0=v(0)=\left(\int_0^1+\int_0^1\right)\ddot{x}(s,z)\,p(s,z)\,ds+p(1,z)\cdot\int_0^1 u(s,z)\,\ddot{x}(s,z)\,ds$$

$$+q(1,z)\,\dot{y}(1,z)$$

$$=2\cdot\int_0^1 p(s,z)\,\ddot{x}(s,z)\,ds+0+1\cdot\dot{x}(0,z),$$

since $q(1,z)=1$ and $\dot{y}(1,z)=\dot{x}(0,z)$. ∎

Now we ask what choice of η_0, ξ_0, μ in Lemma 2.5 yields a solution of the boundary value problem (**) in Lemma 2.3(1). The result is

LEMMA 2.7. *Let* $z>0$, $(\eta_0,\xi_0,\mu)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ *and* η, ξ *be as in Lemma* 2.5, *i.e.*,

$$\dot{\eta}=\alpha(z)\,f'(x(\cdot,z))\xi,\qquad \eta(0)=\eta_0$$

$$\dot{\xi}=-\alpha(z)\,f'(y(\cdot,z))\eta+\mu\ddot{x}(\cdot,z),\qquad \xi(0)=\xi_0,$$

Then the following are equivalent:

(i) $\xi(0)=0$, $\xi\neq 0$, $\eta(1)=\mu\dot{x}(0,z)$, $\eta(0)=\xi(1)$

(ii) $\xi_0=0$, $\eta_0\neq 0$, $\mu=2(\alpha'(z)/\alpha(z))\eta_0$, $\alpha'(z)\int_0^1 u(s,z)\,\ddot{x}(s,z)\,ds=0$.

Proof. (i)$\Rightarrow$(ii): Lemma 2.5(2) gives

$$\xi_0=\xi(0)=0\qquad\text{and}\qquad \eta_0=\eta(0)=\xi(1)=\eta_0-\mu\cdot\int_0^1 u(s,z)\,\ddot{x}(s,z)\,ds,$$

hence,

$$\mu\cdot\int_0^1 u(s,z)\,\ddot{x}(s,z)\,ds=0.$$

It follows that

$$\mu\dot{x}(0,z)=\eta(1)$$

$$=p(1,z)\eta_0-\xi_0-\mu\cdot\left[p(1,z)\int_0^1 u(s,z)\,\ddot{x}(s,z)\,ds\right.$$

$$\left.+\int_0^1 p(s,z)\,\ddot{x}(s,z)\,ds\right]$$

$$=p(1,z)\eta_0-\mu\int_0^1 p(s,z)\,\ddot{x}(s,z)\,ds,$$

and hence

$$p(1, z)\eta_0 = \mu\left[\dot{x}(0, z) + \int_0^1 p(s, z)\,\ddot{x}(s, z)\,ds\right].$$

Case 1. $\mu = 0$: Then Lemma 2.4(4) gives $p(1, z)\eta_0 = 0 = -\alpha'(z) f(z)\eta_0$, resp. $\alpha'(z)\eta_0 = 0$ and $\mu = 0 = 2(\alpha'(z)/\alpha(z))\eta_0$.

Case 2. $\mu \neq 0$: Then $\int_0^1 u(s, z)\,\ddot{x}(s, z)\,ds = 0$ and hence $\alpha'(z)\int_0^1 u(s, z)\,\ddot{x}(s, z)\,ds = 0$. Thus $\int_0^1 p(s, z)\,\ddot{x}(s, z)\,ds = -\frac{1}{2}\dot{x}(0, z)$ by Lemma 2.6, and it follows that $p(1, z)\eta_0 = \mu\cdot\frac{1}{2}\dot{x}(0, z)$ resp. $\mu = 2(p(1, z)/\dot{x}(0, z))\eta_0 = 2(\alpha'(z)/\alpha(z))\eta_0$ by Lemma 2.4(4).

It remains to show $\eta_0 \neq 0$, since we then have $\alpha'(z) = 0$ in Case 1, and hence $\alpha'(z)\int_0^1 u(s, z)\,\ddot{x}(s, z)\,ds = 0$ too. But $\eta_0 = 0$ implies $\mu = 0$, and (i) gives $\xi = \eta = 0$ since $\eta(0) = \xi(0) = 0$. This is a contradiction to $\xi \neq 0$.
(ii) $\Rightarrow$ (i): We have

$$\mu\int_0^1 u(s, z)\,\ddot{x}(s, z)\,ds = 2\frac{\eta_0}{\alpha(z)}\left[\alpha'(z)\int_0^1 u(s, z)\,\ddot{x}(s, z)\,ds\right] = 0,$$

and Lemma 2.5(2) and Lemma 2.6 give

$$\begin{aligned}
\xi(0) &= \xi_0 = 0,\\
\xi(1) &= \eta_0 - \mu\int_0^1 u(s, z)\,\ddot{x}(s, z)\,ds = \eta(0),\\
\eta(1) &= p(1, z)\eta_0 - \xi_0\\
&\quad - \mu\left[p(1, z)\int_0^1 u(s, z)\,\ddot{x}(s, z)\,ds + \int_0^1 p(s, z)\,\ddot{x}(s, z)\,ds\right]\\
&= p(1, z)\eta_0 - \mu\int_0^1 p(s, z)\,\ddot{x}(s, z)\,ds\\
&= p(1, z)\eta_0 + \frac{1}{2}\mu\dot{x}(0, z)\\
&= -\alpha'(z) f(z)\eta_0 + \frac{1}{2}\mu\dot{x}(0, z)\\
&= -\alpha(z) f(z)\cdot\frac{1}{2}\cdot 2\frac{\alpha'(z)}{\alpha(z)}\eta_0 + \frac{1}{2}\mu\dot{x}(0, z)\\
&= \dot{x}(0, z)\cdot\frac{1}{2}\cdot\mu + \frac{1}{2}\mu\dot{x}(0, z) = \mu\dot{x}(0, z),
\end{aligned}$$

and it remains to show $\xi \neq 0$. But $\xi = 0$ yields $\eta_0 = \xi(1) = 0$. ∎

Now let $p(z) = p(\cdot, z)|_{[0,1]}$, etc. Then

LEMMA 2.8. $\dim \ker(U(z)+1_{C_0})=1 \Leftrightarrow \alpha'(z)\cdot\int_0^1 u(s,z)\,\ddot{x}(s,z)\,ds=0$. *If*

$$\xi := q(z)+2\,\frac{\alpha'(z)}{\alpha(z)} \times\left[v(z)\cdot\int_0^{\cdot} p(s,z)\,\ddot{x}(s,z)\,ds - q(z)\cdot\int_0^{\cdot} u(s,z)\,\ddot{x}(s,z)\,ds\right]$$

$$\eta := p(z)+2\,\frac{\alpha'(z)}{\alpha(z)} \times\left[u(z)\cdot\int_0^{\cdot} p(s,z)\,\ddot{x}(s,z)\,ds - p(z)\cdot\int_0^{\cdot} u(s,z)\,\ddot{x}(s,z)\,ds\right]$$

then

(i) $\xi(0)=0,\ \xi(1)=\eta(0)=1,\ \eta(1)=-2\alpha'(z)\,f(z)$,

(ii) $\eta=V(z)\xi$,

(iii) $\ker(U(z)+1_{C_0})=\mathbb{R}\xi$.

Proof. Let $\dim\ker(U(z)+1_{C_0})=1$. Then there is a $\xi\in C_0$ with $U(z)\xi=-\xi$ and $\xi\neq 0$. Define $\eta:=V(z)\xi$, $\mu:=\eta(1)/\dot{x}(0,z)$, then $\eta(0)=\xi(1)$, and (**) holds. Hence Lemma 2.7(i) is true and Lemma 2.7(ii) gives

$$\alpha'(z)\cdot\int_0^1 u(s,z)\,\ddot{x}(s,z)\,ds=0.$$

Now suppose $\alpha'(z)\cdot\int_0^1 u(s,z)\,\ddot{x}(s,z)\,ds=0$. We prove

(a) $\ker(U(z)+1_{C_0})\neq\{0\}$,

(b) if $\xi_1,\ \xi_2\in\ker(U(z)+1_{C_0})$, then $\xi_1,\ \xi_2$ are linearly dependent, from which $\dim\ker(U(z)+1_{C_0})=1$ follows.

Proof of (a): Let $\eta_0:=1$, $\xi_0:=0$, $\mu:=2(\alpha'(z)/\alpha(z))$ and (η,ξ) be as in Lemma 2.5(1), i.e.,

$$\eta=p(\cdot,z)+2\,\frac{\alpha'(z)}{\alpha(z)} \times\left[u(\cdot,z)\cdot\int_0^{\cdot} p(s,z)\,\ddot{x}(s,z)\,ds - p(\cdot,z)\cdot\int_0^{\cdot} u(s,z)\,\ddot{x}(s,z)\,ds\right],$$

$$\xi=q(\cdot,z)+2\,\frac{\alpha'(z)}{\alpha(z)} \times\left[v(\cdot,z)\cdot\int_0^{\cdot} p(s,z)\,\ddot{x}(s,z)\,ds - q(\cdot,z)\cdot\int_0^{\cdot} u(s,z)\,\ddot{x}(s,z)\,ds\right].$$

Then Lemma 2.7(ii) is true. Now Lemma 2.7(i) and Lemma 2.3(2) give $\xi \neq 0$ and $U(z)\xi = -\xi$. Hence $\xi \in \ker(U(z) + 1_{C_0})$ and $\ker(U(z) + 1_{C_0}) \neq \{0\}$. Moreover Lemma 2.7(i) and $\dot{x}(0, z) = -\alpha(z) f(z)$ show (i) and (ii).

Proof of (b) *and hence of* (iii): Let $\xi_1, \xi_2 \in \ker(U(z) + 1_{C_0})$. If $\xi_1 = 0$ or $\xi_2 = 0$, the assumption is true. Suppose now $\xi_1 \neq 0$, $\xi_2 \neq 0$ and

$$\eta_1 := V(z)\xi_1, \qquad \eta_2 := V(z)\xi_2, \qquad \mu_1 := \frac{\eta_1(z)}{\dot{x}(0, z)}, \qquad \mu_2 := \frac{\eta_2(z)}{\dot{x}(0, z)}.$$

Then (η_1, ξ_1, μ_1) and (η_2, ξ_2, μ_2) fulfil the equations in Lemma 2.3(1). In particular $\eta_i(0) = \xi_i(1)$ for $i = 1, 2$, and Lemma 2.7 gives $\mu_i = 2(\alpha'(z)/\alpha(z))\,\eta_i(0)$, $i = 1, 2$. Since (**) is linear in η, ξ, μ, (**) is also fulfilled for every linear combination of (η_1, ξ_1, μ_1) and (η_2, ξ_2, μ_2). We set $c_1 := \xi_2(1)$, $c_2 := -\xi_1(1)$ and

$$\xi := c_1\xi_1 + c_2\xi_2, \qquad \eta := c_1\eta_1 + c_2\eta_2, \qquad \mu := c_1\mu_1 + c_2\mu_2;$$

then

$$\begin{aligned} \dot{\eta} &= \alpha f'(x(\cdot, z))\xi, & \eta(0) &= \xi(1) \\ \dot{\xi} &= -\alpha f'(y(\cdot, z))\eta + \mu\ddot{x}(\cdot, z), & \xi(0) &= 0. \end{aligned}$$

But $\eta(0) = c_1\eta_1(0) + c_2\eta_2(0) = c_1\xi_1(1) + c_2\xi_2(1) = 0$, and analogously $\mu = 0$, hence,

$$\begin{aligned} \dot{\eta} &= \alpha f'(x)\xi, & \eta(0) &= 0 \\ \dot{\xi} &= -\alpha f'(y)\eta, & \xi(0) &= 0. \end{aligned}$$

From this we find $\eta = 0$, $\xi = c_1\xi_1 + c_2\xi_2 = 0$, and ξ_1, ξ_2 are linearly dependent if $(c_1, c_2) \neq (0, 0)$. If $c_1 = \xi_2(1) = 0$ then $\eta_2(0) = \xi_2(1) = 0$, $\mu_2 = 2(\alpha'(z)/\alpha(z))\,\eta_2(0) = 0$, and (**) for (η_2, ξ_2, μ_2) gives $\eta_2 = \xi_2 = 0$. ∎

Thus we have the final result of this section:

THEOREM 2.1. *Let $f \in C^2(\mathbb{R})$ be odd, $f(0) = 0$, $f'(0) < 0$, $xf(x) < 0$ for $x \neq 0$. Let $z > 0$. Then for the operator $U(z)$ the following hold*:

(1) *If $\alpha'(z) = 0$, then* $\dim\ker(U(z) + 1_{C_0}) = 1$ *and* $\ker(U(z) + 1_{C_0}) = \mathbb{R}q(z)$.

(2) *If $\alpha'(z) \neq 0$, then* $\dim\ker(U(z) + 1_{C_0}) = 1$ *if and only if* $\int_0^1 u(s, z)\,\ddot{x}(s, z)\,ds = 0$. *In this case* $\ker(U(z) + 1_{C_0}) = \mathbb{R}\xi$ *where ξ is as in Lemma* 2.8.

3. The Bifurcation Theorem

Define $r(z) := \int_0^1 u(t, z)\,\ddot{x}(t, z)\,dt$ for $z > 0$. Then by the Theorems 1.3 and 2.1 the conditions for $z^* > 0$ to be a differentiable bifurcation point are:

(1) $\alpha'(z^*) \neq 0$,

(2) $r(z^*) = 0$,

(3) $\chi^* := U'(z^*)\xi^* \notin \operatorname{im}(U(z^*) + 1_{C_0})$,

(4) $U(z^*) + 1_{C_0}$ is a Fredholm operator with index 0,

with $\xi^* \in \ker(U(z^*) + 1_{C_0})$ of Lemma 2.8.

In this section we assume $\alpha'(z^*) \neq 0$ and $r(z^*) = 0$ for some $z^* > 0$. We shall now study the conditions (3) and (4). To this end we describe $\operatorname{im}(U(z^*) + 1_{C_0})$ as the kernel of a linear map $L: C_0 \to \mathbb{R}$. First

LEMMA 3.1. *Let $\chi \in C_0$, $\mu \in \mathbb{R}$, $\kappa \in \mathbb{R}$ and let $\binom{\psi}{v}$ be the solution of*

$$\begin{pmatrix} \dot\psi \\ \dot v \end{pmatrix} = A(\cdot, z^*)\begin{pmatrix} \psi \\ v \end{pmatrix} + \begin{pmatrix} \alpha(z^*)\, f'(x(\cdot, z^*))\chi \\ \mu \ddot{x}(\cdot, z^*) \end{pmatrix}, \qquad \begin{pmatrix} \psi(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} \kappa \\ 0 \end{pmatrix}.$$

Then:

$$\psi(1) = p(1, z^*)\, v(1) + \frac{1}{2}\mu \dot{x}(0, z^*) + \int_0^1 \dot{p}(s, z^*)\,\chi(s)\,ds,$$

$$v(1) = \kappa + \int_0^1 \dot{u}(s, z^*)\,\chi(s)\,ds.$$

Proof. Since we know the fundamental solution of the corresponding homogenous system (*), we can solve (*) immediately. Definition 2.1, Remark 2.1, and Lemma 2.4(5) yield

$$\begin{aligned}
\begin{pmatrix} \psi \\ v \end{pmatrix} &= S(\cdot, z^*)\left[\begin{pmatrix} \kappa \\ 0 \end{pmatrix} + \int_0^{\cdot} S(s, z^*)^{-1}\begin{pmatrix} \alpha(z^*)\, f'(x(s, z^*))\,\chi(s) \\ \mu \ddot{x}(s, z^*) \end{pmatrix} ds\right] \\
&= S(\cdot, z^*)\left[\begin{pmatrix} \kappa \\ 0 \end{pmatrix} + \int_0^{\cdot} \begin{pmatrix} v & -u \\ -q & p \end{pmatrix}\begin{pmatrix} \alpha f'(x)\chi \\ \mu \ddot{x} \end{pmatrix} ds\right] \\
&= S(\cdot, z^*)\left[\begin{pmatrix} \kappa \\ 0 \end{pmatrix} + \int_0^{\cdot} \begin{pmatrix} \alpha f'(x) v \\ -\alpha f'(x) q \end{pmatrix} \chi\, ds + \int_0^{\cdot} \mu \begin{pmatrix} -u\ddot{x} \\ p\ddot{x} \end{pmatrix} ds\right] \\
&= S(\cdot, z^*)\left[\begin{pmatrix} \kappa \\ 0 \end{pmatrix} + \int_0^{\cdot} \begin{pmatrix} \dot{u}\chi \\ -\dot{p}\chi \end{pmatrix} ds + \mu \int_0^{\cdot} \begin{pmatrix} -u\ddot{x} \\ p\ddot{x} \end{pmatrix} ds\right].
\end{aligned}$$

Now Lemma 2.4(4) and Lemma 2.6 give

$$\begin{pmatrix}\psi(1)\\ \nu(1)\end{pmatrix}=\begin{pmatrix}p(1,z^*) & -1\\ 1 & 0\end{pmatrix}\left[\begin{pmatrix}\kappa\\ 0\end{pmatrix}+\int_0^1\begin{pmatrix}\dot u\chi\\ -\dot p\chi\end{pmatrix}ds+\mu\begin{pmatrix}-r(z^*)\\ -\frac{1}{2}\dot x(0,z^*)\end{pmatrix}\right]$$
$$=\begin{pmatrix}p(1,z^*) & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\kappa+\int_0^1\dot u\chi\,ds\\ -\int_0^1\dot p\chi\,ds-\frac{1}{2}\mu\dot x(0,z^*)\end{pmatrix},$$

from which our assumption follows. ∎

LEMMA 3.2. *Let* $L\colon C_0\ni\chi\mapsto\chi(1)+\int_0^1\dot u(s,z^*)\,\chi(s)\,ds\in\mathbb{R}$. *L is linear and continuous, and* $\operatorname{im}(U(z^*)+1_{C_0})=\ker L$.

Proof. We only show: $\chi\in\operatorname{im}(U(z^*)+1_{C_0})\Leftrightarrow L\chi=0$ for all $\chi\in C_0$. Let $\chi\in\operatorname{im}(U(z^*)+1_{C_0})$, i.e., $\chi=U(z^*)\varphi+\varphi$ for some $\varphi\in C_0$. Define $\nu:=\varphi-\chi$, $\psi:=V(z^*)\varphi$ and $\mu:=\psi(1)/\dot x(0,z^*)$, $\kappa:=\psi(0)$. Then by Lemma 2.1(1, 3) we have

$$\dot\psi=\alpha f'(x)\varphi=\alpha f'(x)\nu+\alpha f'(x)\chi,\qquad \psi(0)=\varphi(1),$$
$$\nu=\varphi-\chi=\varphi-(U(z^*)\varphi+\varphi)=-U(z^*)\varphi$$
$$=-W(z^*)\varphi+\dot x(\cdot,z^*)\frac{[W(z^*)\varphi](0)}{\dot x(0,z^*)}.$$

This shows that ν is differentiable, and Lemma 2.1(2) gives $[W(z^*)\varphi](0)=[V(z^*)\varphi](1)=\psi(1)$. Because of $\varphi,\chi\in C_0$ we thus have

$$\dot\nu=-(W(z^*)\varphi)^{\cdot}+\mu\ddot x(\cdot,z^*)=-\alpha f'(y)\psi+\mu\ddot x,\qquad \nu(0)=\varphi(0)-\chi(0)=0,$$

or

$$\begin{pmatrix}\dot\psi\\ \dot\nu\end{pmatrix}=\begin{pmatrix}0 & \alpha f'(x)\\ -\alpha f'(y) & 0\end{pmatrix}\begin{pmatrix}\psi\\ \nu\end{pmatrix}+\begin{pmatrix}\alpha f'(x)\chi\\ \mu\ddot x\end{pmatrix},\qquad \begin{pmatrix}\psi(0)\\ \nu(0)\end{pmatrix}=\begin{pmatrix}\kappa\\ 0\end{pmatrix}.$$

The previous lemma shows

$$\kappa-\chi(1)=\psi(0)-\chi(1)=\varphi(1)-\chi(1)=\nu(1)$$
$$=\kappa+\int_0^1\dot u(s,z^*)\,\chi(s)\,ds,$$

and hence $L\chi=0$.

Now suppose $L\chi=0$. Define

$$\kappa:=0,\qquad \mu:=\frac{2}{\dot x(0,z^*)}\left[-p(1,z^*)\,\chi(1)+\int_0^1\dot p(s,z^*)\,\chi(s)\,ds\right],$$

and let $\binom{\psi}{v}$ be as in Lemma 3.1. Then we have for $\varphi := v + \chi$:

$$\psi(1) = p(1, z^*)\left[\kappa + \int_0^1 \dot{u}\chi\, ds\right] + \frac{1}{2}\mu\dot{x}(0, z^*) + \int_0^1 \dot{p}\chi\, ds$$

$$= \frac{1}{2}\mu\dot{x}(0, z^*) + \left[-p(1, z^*)\,\chi(1) + \int_0^1 \dot{p}\chi\, ds\right]$$

$$= \frac{1}{2}\mu\dot{x}(0, z^*) + \frac{1}{2}\mu\dot{x}(0, z^*) = \mu\dot{x}(0, z^*),$$

$$\varphi(0) = v(0) + \chi(0) = 0 + 0 = 0,$$

$$\varphi(1) = v(1) + \chi(1) = \kappa + \int_0^1 \dot{u}(s, z^*)\,\chi(s)\, ds + \chi(1)$$

$$= 0 + L\chi = 0 = \psi(0),$$

$$\dot{\psi} = \alpha f'(x)\,v + \alpha f'(x)\,\chi = \alpha f'(x)\,\varphi.$$

Hence Lemma 2.1(1) shows $V(z^*)\varphi = \psi$, and Lemma 2.1(2) gives

$$W(z^*)\varphi = \psi(1) + \int_0^{\cdot} \alpha f'(y)\psi\, ds = \psi(1) + \int_0^{\cdot} (\mu\ddot{x} - \dot{v})\, ds$$

$$= \psi(1) + (\mu\dot{x} - v) - (\mu\dot{x}(0, z^*) - v(0))$$

$$= \mu\dot{x} - v + (\psi(1) - \mu\dot{x}(0, z^*)) = \mu\dot{x} - v.$$

Since $[W(z^*)\varphi](0) = [V(z^*)\varphi](1) = \psi(1) = \mu\dot{x}(0, z^*)$, we obtain

$$U(z^*)\varphi = W(z^*)\varphi - \dot{x}\,\frac{[W(z^*)\varphi](0)}{\dot{x}(0, z^*)} = \mu\dot{x} - v - \mu\dot{x} = -v = -\varphi + \chi,$$

and thus $\chi = U(z^*)\varphi + \varphi \in \operatorname{im}(U(z^*) + 1_{C_0})$.

COROLLARY 3.1. *$U(z^*) + 1_{C_0}$ is a Fredholm operator with index* 0.

Proof. $\operatorname{im}(U(z^*) + 1_{C_0}) = \ker L$ is closed, $\dim\ker(U(z^*) + 1_{C_0}) = 1$, and it remains to show $\operatorname{codim}\operatorname{im}(U(z^*) + 1_{C_0}) = \operatorname{codim}\ker L = 1$. But $\operatorname{codim}\ker L = 1 \Leftrightarrow \operatorname{im} L = \mathbb{R} \Leftrightarrow L$ is surjective $\Leftrightarrow L\chi \neq 0$ for some $\chi \in C_0$. Since $u(z^*) = u(\cdot, z^*)|_{[0,1]} \in C_0$ and $u(1, z^*) = -1$ we have

$$Lu(z^*) = u(z^*)(1) + \int_0^1 \dot{u}(s, z^*)\, u(s, z^*)\, ds$$

$$= u(1, z^*) + \frac{1}{2}u^2(\cdot, z^*)|_0^1 = u(1, z^*) + \frac{1}{2}u^2(1, z^*) = -\frac{1}{2}. \quad \blacksquare$$

Now $\chi^* = U'(z^*)\xi^* \notin \operatorname{im}(U(z^*) + 1_{C_0})$ is the last condition we have to study. By Lemma 3.2 it is equivalent to $L\chi^* \neq 0$. For $L\chi^*$ we find

LEMMA 3.3. *$L\chi^* = \mu^* r'(z^*)$ where $\mu^* := 2(\alpha'(z^*)/\alpha(z^*))$.*

Proof. (1) Define $\xi: \mathbb{R}^+ \to C_0$, $\eta: \mathbb{R}^+ \to C$, $\zeta: \mathbb{R}^+ \to C_0$ and $\mu: \mathbb{R}^+ \to \mathbb{R}$ by $\xi(z) := \xi^*$, $\eta(z) := V(z)\xi^*$, $\zeta(z) := U(z)\xi^*$, $\mu(z) := \eta(z)(1)/\dot{x}(0, z)$ for $z > 0$. Furthermore let

$$\xi(t, z) := \xi(z)(t) = \xi^*(t), \qquad \eta(t, z) := \eta(z)(t), \qquad \zeta(t, z) := \zeta(z)(t)$$

and

$$g(t, z) := \alpha(z)\, f'(x(t, z)), \qquad h(t, z) := \alpha(z)\, f'(y(t, z))$$

for $(t, z) \in [0, 1] \times \mathbb{R}^+$. Then by Lemma 2.1

$$\begin{aligned} \dot{\eta} &= g\xi, & \eta(0, z) &= \xi(1, z) = \xi^*(1) \\ \dot{\zeta} &= h\eta - \mu\ddot{x}, & \eta(1, z) &= [W(z)\xi^*](0) = \mu(z)\, \dot{x}(0, z). \end{aligned}$$

In particular we have for $z = z^*$ (compare Lemma 2.8(i))

$$\xi(z^*) = \xi^*, \qquad \eta(z^*) = V(z^*)\xi^* =: \eta^*,$$

$$\zeta(z^*) = U(z^*)\xi^* = -\xi^*, \qquad \mu(z^*) = \frac{\eta^*(1)}{\dot{x}(0, z^*)} = \mu^*,$$

and if $g^* := g(\cdot, z^*)$, $h^* := h(\cdot, z^*)$, $\ddot{x}^* := \ddot{x}(\cdot, z^*)$ then

$$\begin{aligned} \dot{\eta}^* &= g^*\xi^*, & \eta^*(0) &= \xi^*(1) \\ \dot{\xi}^* &= -h^*\eta^* + \mu^*\ddot{x}^*, & \eta^*(1) &= \mu^*\dot{x}(0, z^*). \end{aligned}$$

(2) Some remarks concerning our notations. In the following u_z, x_z is the partial derivative of u, x with respect to z, $d/dz|_{z^*}$, $\partial/\partial z|_{z^*}$ denotes the derivative with respect to z, evaluated at $z = z^*$. Furthermore let $x^* := x(\cdot, z^*)$, $u_z^* := u_z(\cdot, z^*) = \partial/\partial z|_{z^*} u$, etc.

(3) Hence we have

$$\chi^* = U'(z^*)\xi^* = \frac{d}{dz}\bigg|_{z^*} U(z)\xi^* = \zeta'(z^*) = \frac{\partial}{\partial z}\bigg|_{z^*} \zeta(\cdot, z) = \zeta_z(\cdot, z^*) = \zeta_z^*;$$

χ^* is differentiable and

$$\dot{\chi}^*(t) = \frac{\partial}{\partial z}\bigg|_{z^*} \dot{\zeta}(t, z) = \frac{\partial}{\partial z}\bigg|_{z^*} [h(t, z)\, \eta(t, z) - \mu(z)\, \ddot{x}(t, z)].$$

Partial integration yields:

$$L\chi^* = \chi^*(1) + \int_0^1 \dot{u}^*\chi^*\,dt = \chi^*(1) + u^*\chi^*|_0^1 - \int_0^1 u^*\dot{\chi}^*\,dt$$

$$= \chi^*(1) + (-1)\cdot\chi^*(1) - 0\cdot\chi^*(0) - \int_0^1 u^*\dot{\chi}^*\,dt = -\int_0^1 u^*\dot{\chi}^*\,dt$$

$$= -\int_0^1 u^* \left.\frac{\partial}{\partial z}\right|_{z^*} [h\eta - \mu\ddot{x}]\,dt$$

$$= -\int_0^1 [u^*h_z^*\eta^* + u^*h^*\eta_z^*]\,dt + \mu'(z^*)\int_0^1 u^*\ddot{x}^*\,dt$$

$$+ \mu(z^*)\int_0^1 u^*\ddot{x}_z^*\,dt$$

$$= -\int_0^1 [u^*h_z^*\eta^* + u^*h^*\eta_z^*]\,dt + \mu'(z^*)\,r(z^*) + \mu^*\int_0^1 u^*\ddot{x}_z^*\,dt$$

$$+ \mu^*\int_0^1 u_z^*\ddot{x}^*\,dt - \mu^*\int_0^1 u_z^*\ddot{x}^*\,dt$$

$$= -\int_0^1 [u^*h_z^*\eta^* + u^*h^*\eta_z^*]\,dt + 0 + \mu^*\int_0^1 \left.\frac{\partial}{\partial z}\right|_{z^*} (u\ddot{x})\,dt$$

$$- \mu^*\int_0^1 u_z^*\ddot{x}^*\,dt$$

$$= -\int_0^1 u^*h_z^*\eta^*\,dt - \int_0^1 u^*h^*\eta_z^*\,dt - \mu^*\int_0^1 u_z^*\ddot{x}^*\,dt + \mu^* r'(z^*).$$

Let $c_1 := \int_0^1 u^*h_z^*\eta^*\,dt$, $c_2 := \int_0^1 u^*h^*\eta_z^*\,dt$, $c_3 := \mu^*\int_0^1 u_z^*\ddot{x}^*\,dt$. Then it remains to prove $c_1 + c_2 + c_3 = 0$.

(4) Remark 2.1 gives $h^*(t)\,u^*(t) = h(t, z^*)\,u(t, z^*) = \alpha(z^*) f'(y(t, z^*))\,u(t, z^*) = -\dot{v}(t, z^*) = -\dot{v}^*(t)$, and since $v(1, z) = 0$, $\eta(0, z) = \xi^*(1)$, partial integration shows

$$c_2 = \int_0^1 u^*h^*\eta_z^*\,dt = \int_0^1 -\dot{v}^*\eta_z^*\,dt = -v^*\eta_z^*|_0^1 + \int_0^1 v^*\dot{\eta}_z^*\,dt$$

$$= -v(1, z^*)\,\eta_z(1, z^*) + v(0, z^*)\left.\frac{d}{dz}\right|_{z^*} \eta(0, z) + \int_0^1 v^*\dot{\eta}_z^*\,dt$$

$$= \int_0^1 v^*\dot{\eta}_z^*\,dt = \int_0^1 v^*\left[\left.\frac{\partial}{\partial z}\right|_{z^*} \dot{\eta}\right]dt = \int_0^1 v^*\left[\left.\frac{\partial}{\partial z}\right|_{z^*} g\xi\right]dt$$

$$= \int_0^1 v^*g_z^*\xi^*\,dt.$$

It follows that

$$c_1+c_2+c_3=\int_0^1 (u^*h_z^*\eta^*+v^*g_z^*\xi^*+\mu^*u_z^*\ddot{x}^*)\,dt.$$

(5) Differentiation of $\dot{u}=gv$, $\dot{v}=-hu$ (compare Remark 2.1 and the definition of h and g) with respect to z gives at $z=z^*$:

$$\dot{u}_z^*=g_z^*v^*+g^*v_z^*,\qquad \dot{v}_z^*=-h_z^*u^*-h^*u_z^*$$

resp.

$$g_z^*v^*=\dot{u}_z^*-g^*v_z^*,\qquad h_z^*u^*=-\dot{v}_z^*-h^*u_z^*.$$

From $\dot{\eta}^*=g^*\xi^*$, $\dot{\xi}^*=-h^*\eta^*+\mu^*\ddot{x}^*$ we derive

$$\begin{aligned}
c_1+c_2+c_3&=\int_0^1 (u^*h_z^*\eta^*+v^*g_z^*\xi^*+\mu^*u_z^*\ddot{x}^*)\,dt\\
&=\int_0^1 [(-\dot{v}_z^*-h^*u_z^*)\eta^*+(\dot{u}_z^*-g^*v_z^*)\xi^*+\mu^*u_z^*\ddot{x}^*]\,dt\\
&=\int_0^1 [-\dot{v}_z^*\eta^*-u_z^*(h^*\eta^*)+\dot{u}_z^*\xi^*-v_z^*(g^*\xi^*)+\mu^*u_z^*\ddot{x}^*]\,dt\\
&=\int_0^1 [-\dot{v}_z^*\eta^*-u_z^*(\mu^*\ddot{x}^*-\dot{\xi}^*)+\dot{u}_z^*\xi^*-v_z^*\dot{\eta}^*+\mu^*u_z^*\ddot{x}^*]\,dt\\
&=\int_0^1 [(-\dot{v}_z^*\eta^*-v_z^*\dot{\eta}^*)+(u_z^*\dot{\xi}^*+\dot{u}_z^*\xi^*)]\,dt\\
&=\int_0^1 [-(v_z^*\eta^*)^{\cdot}+(u_z^*\xi^*)^{\cdot}]\,dt=[u_z^*\xi^*-v_z^*\eta^*]\big|_0^1\\
&=[u_z(1,z^*)\,\xi^*(1)-v_z(1,z^*)\,\eta^*(1)]\\
&\quad-[u_z(0,z^*)\,\xi^*(0)-v_z(0,z^*)\,\eta^*(0)]\\
&=0,
\end{aligned}$$

since $u(0,z)=0$, $v(0,z)=1$, $u(1,z)=-1$, $v(1,z)=0$ and hence $u_z(1,z^*)=v_z(1,z^*)=u_z(0,z^*)=v_z(0,z^*)=0$. ∎

In the next lemma we shall calculate $r'(z)$. For this purpose we need some preliminary remarks.

(i) Let $\alpha=\alpha(z)$ and $b:=f'(0)$. Then differentiation of

$$\begin{aligned}
\dot{x}&=\alpha f(y), & x(0,z)&=0, & x(1,z)&=z\\
\dot{y}&=-\alpha f(x), & y(0,z)&=-z, & y(1,z)&=0,
\end{aligned}$$

gives

$$\ddot{x} = \alpha f'(y)\dot{y} = -\alpha^2 f'(y) f(x), \qquad \dot{x}(0,z) = -\alpha f(z), \qquad \dot{x}(1,z) = 0$$

$$\ddot{y} = -\alpha f'(x)\dot{x} = -\alpha^2 f'(x) f(y), \qquad \dot{y}(0,z) = 0, \qquad \dot{y}(1,z) = -\alpha f(z)$$

$$\dddot{x} = \alpha^3 f''(y) f^2(x) - \alpha^3 f'(x) f'(y) f(y), \qquad \ddot{x}(0,z) = 0, \qquad \ddot{x}(1,z) = -\alpha^2 f(z)\cdot b$$

$$\dddot{y} = -\alpha^3 f''(x) f^2(y) + \alpha^3 f'(x) f'(y) f(x), \qquad \ddot{y}(0,z) = \alpha^2 f(z)\cdot b, \qquad \ddot{y}(1,z) = 0.$$

(ii) From Lemma 2.4(1) we see

$$r(z) = \int_0^1 u(t,z)\,\ddot{x}(t,z)\,dt = \frac{\alpha^2(z)}{f(z)}\int_0^1 f^2(x(t,z))\,f'(y(t,z))\,dt$$

$$= \frac{1}{\alpha f(z)}\int_0^1 \ddot{x}(t,z)\,\dot{y}(t,z)\,dt.$$

Theorem 1.1 yields

$$x(1-t,z) = -x(t-1,z) = -y(t,z), \qquad y(1-t,z) = x(-t,z) = -x(t,z)$$

for $z > 0$, $t \in \mathbb{R}$, and the substitution $t \mapsto 1-t$ in $r(z)$ gives (observe that f is odd)

$$r(z) = \frac{\alpha^2(z)}{f(z)}\int_0^1 f^2(y(t,z))\,f'(x(t,z))\,dt$$

$$= -\frac{1}{\alpha f(z)}\int_0^1 \dot{x}(t,z)\,\ddot{y}(t,z)\,dt,$$

and hence:

$$r(z) = \frac{1}{2}\frac{\alpha^2(z)}{f(z)}\int_0^1 [f^2(x)\,f'(y) + f^2(y)\,f'(x)]\,dt$$

$$= \frac{1}{2\alpha f(z)}\int_0^1 (\ddot{x}\dot{y} - \dot{x}\ddot{y})\,dt.$$

From now on we omit the arguments (t, z).

The last two integrals are invariant under $t \mapsto 1-t$.

(iii) By Lemma 2.4(1, 5) we have $pv - qu = 1$ and $\alpha f(z)u = \dot{y}$, $\alpha f(z)v = -\dot{x}$. It follows that $\alpha(z)\,f(z) = -p\dot{x} - q\dot{y}$ resp. $f(z) = -pf(y) + qf(x)$.

(iv) Let $F := \int_0^{\cdot} f(u)\,du$. Then by Theorem 1.1(3) $F(x) + F(y) = F(z)$, and differentiation with respect to z yields $f(x)x_z + f(y)\,y_z = f(z)$.

LEMMA 3.4. $r'(z) = (\alpha'(z)/\alpha(z) - f'(z)/f(z))\, r(z) + \alpha^2(z) \cdot J(z)$ *where* $J(z) := \int_0^1 f'(x(t, z))\, f'(y(t, z))\, dt$.

Proof. (1) Let $a(z) := \frac{1}{2}(\alpha^2(z)/f(z))$. Then

$$r(z) = a(z) \cdot \int_0^1 [f^2(x)\, f'(y) + f^2(y)\, f'(x)]\, dt$$

and

$$\begin{aligned}
r'(z) &= \frac{a'(z)}{a(z)} r(z) + a(z) \int_0^1 \frac{\partial}{\partial z} [f^2(x)\, f'(y) + f^2(y)\, f'(x)]\, dt \\
&= \frac{a'(z)}{a(z)} r(z) + a(z) \int_0^1 [2f(x)\, f'(x)\, f'(y) + f^2(y)\, f''(x)]\, x_z\, dt \\
&\quad + a(z) \int_0^1 [f^2(x)\, f''(y) + 2f(y)\, f'(x)\, f'(y)]\, y_z\, dt \\
&= \frac{a'(z)}{a(z)} r(z) + a(z) \int_0^1 \left[-\frac{1}{\alpha^3} \ddot{y} + 3f(x)\, f'(x)\, f'(y) \right] x_z\, dt \\
&\quad + a(z) \int_0^1 \left[\frac{1}{\alpha^3} \ddot{x} + 3f'(x)\, f'(y)\, f(y) \right] y_z\, dt \\
&= \frac{a'(z)}{a(z)} r(z) + \frac{a(z)}{\alpha^3} \int_0^1 (\ddot{x} y_z - \ddot{y} x_z)\, dt \\
&\quad + 3a(z) \int_0^1 [f(x) x_z + f(y)\, y_z]\, f'(x)\, f'(y)\, dt \\
&= \frac{a'(z)}{a(z)} r(z) + \frac{a(z)}{\alpha^3} \int_0^1 (\ddot{x} y_z - \ddot{y} x_z)\, dt \\
&\quad + 3a(z)\, f(z) \cdot \int_0^1 f'(x)\, f'(y)\, dt.
\end{aligned}$$

(2) Calculation of $d := \int_0^1 (\ddot{x} y_z - \ddot{y} x_z)\, dt$: From Lemma 2.4(2) we have

$$x_z = q - \alpha' f(z) \cdot t \cdot v = q + \frac{\alpha'}{\alpha} t\dot{x},$$

$$y_z = -p + \alpha' f(z) \cdot t \cdot u = -p + \frac{\alpha'}{\alpha} t\dot{y}$$

and hence

$$d=\int_0^1\left(-\bar{x}p+\frac{\alpha'}{\alpha}t\bar{x}\dot{y}-\bar{y}q-\frac{\alpha'}{\alpha}t\bar{y}\dot{x}\right)dt=-d_1+\frac{\alpha'}{\alpha}\cdot d_2$$

with

$$d_1:=\int_0^1(\bar{x}p+\bar{y}q)\,dt,\qquad d_2:=\int_0^1 t\cdot(\bar{x}\dot{y}-\bar{y}\dot{x})\,dt.$$

Calculation of d_1: Since $\dot{p}=\alpha f'(x)q$, $\dot{q}=-\alpha f'(y)p$ it follows that

$$\begin{aligned}
d_1&=\int_0^1\left[p\left(\frac{d}{dt}\ddot{x}\right)+q\left(\frac{d}{dt}\ddot{y}\right)\right]dt\\
&=(p\ddot{x}+q\ddot{y})|_0^1-\int_0^1(\dot{p}\ddot{x}+\dot{q}\ddot{y})\,dt\\
&=(p(1,z)\,\ddot{x}(1,z)+q(1,z)\,\ddot{y}(1,z))-(p(0,z)\,\ddot{x}(0,z)+q(0,z)\,\ddot{y}(0,z))\\
&\quad-\int_0^1[\alpha f'(x)\,q\ddot{x}-\alpha f'(y)\,p\ddot{y}]\,dt\\
&=(-\alpha^2 f(z)\,bp(1,z)+0)-(0+0)\\
&\quad-\int_0^1[-\alpha f'(x)\,q\alpha^2 f'(y)\,f(x)+\alpha f'(y)\,p\alpha^2 f'(x)\,f(y)]\,dt\\
&=\alpha^2\alpha' f^2(z)b-\alpha^3\cdot\int_0^1 f'(x)\,f'(y)\,[pf(y)-qf(x)]\,dt\\
&=\alpha^2\alpha' f^2(z)b-\alpha^3\cdot\int_0^1 f'(x)\,f'(y)\,[-f(z)]\,dt\\
&=\alpha^2\alpha' f^2(z)b+\alpha^3 f(z)\cdot\int_0^1 f'(x)\,f'(y)\,dt.
\end{aligned}$$

Calculation of d_2:

$$\begin{aligned}
d_2&=\int_0^1 t\cdot(\bar{x}\dot{y}-\bar{y}\dot{x})\,dt=\int_0^1 t\cdot\frac{d}{dt}(\ddot{x}\dot{y}-\ddot{y}\dot{x})\,dt\\
&=t\cdot(\ddot{x}\dot{y}-\ddot{y}\dot{x})|_0^1-\int_0^1\frac{dt}{dt}(\ddot{x}\dot{y}-\ddot{y}\dot{x})\,dt\\
&=\ddot{x}(1,z)\,\dot{y}(1,z)-\ddot{y}(1,z)\,\dot{x}(1,z)-\int_0^1(\ddot{x}\dot{y}-\ddot{y}\dot{x})\,dt\\
&=(-\alpha^2 f(z)b)\cdot(-\alpha f(z))-0\cdot 0-2\alpha f(z)\,r(z)\\
&=\alpha^3 f^2(z)b-2\alpha f(z)\,r(z).
\end{aligned}$$

Altogether we find

$$d = -d_1 + \frac{\alpha'}{\alpha} d_2$$

$$= -\alpha^2\alpha' f^2(z) b - \alpha^3 f(z) \int_0^1 f'(x)\, f'(y)\, dt$$

$$+ \frac{\alpha'}{\alpha} \cdot \alpha^3 f^2(z) b - \frac{\alpha'}{\alpha} \cdot 2\alpha f(z)\, r(z)$$

$$= -2\alpha' f(z)\, r(z) - \alpha^3 f(z) \int_0^1 f'(x)\, f'(y)\, dt.$$

(3) Thus we have

$$r'(z) = \frac{a'(z)}{a(z)} r(z) + \frac{a(z)}{\alpha^3} \left[-2\alpha' f(z)\, r(z) - \alpha^3 f(z) \int_0^1 f'(x)\, f'(y)\, dt \right]$$

$$+ 3a(z)\, f(z) \int_0^1 f'(x)\, f'(y)\, dt$$

$$= \left[\frac{a'(z)}{a(z)} - 2a(z) \frac{\alpha'}{\alpha^3} f(z) \right] r(z) + 2a(z)\, f(z) \cdot \int_0^1 f'(x)\, f'(y)\, dt,$$

and differentiation of $a = \frac{1}{2}(\alpha^2/f)$ yields the expression for $r'(z)$ stated above. ∎

By Lemma 3.3, $\mu^* = 2\alpha'(z^*)/\alpha(z^*)$, and it immediately follows

COROLLARY 3.2. $L\chi^* = 2\alpha(z^*)\, \alpha'(z^*)\, J(z^*)$.

Now Theorem 1.3, Theorem 2.1, Corollary 3.1, and Corollary 3.2 yield

THEOREM 3.1 (Bifurcation of symmetric periodic solutions). *Let $f \in C^2(\mathbb{R})$ be odd, $f(0) = 0$, $f'(0) < 0$, and $xf(x) < 0$ for $x \neq 0$. Let $z > 0$ and $x : \mathbb{R} \to \mathbb{R}$ be the symmetric period-4 solution of*

$$\dot{x}(t) = \alpha(z)\, f(x(t-1))$$

with $x(0) = 0$ and amplitude $x(1) = z$. Let $y := x(\cdot - 1)$ and suppose that

(1) $\alpha'(z) \neq 0$

(2) $\int_0^1 f^2(x)\, f'(y)\, dt = 0$

(3) $\int_0^1 f'(x)\, f'(y)\, dt \neq 0$.

Then z is a differentiable bifurcation point where symmetric periodic solutions bifurcate from the Kaplan-Yorke-branch of f.

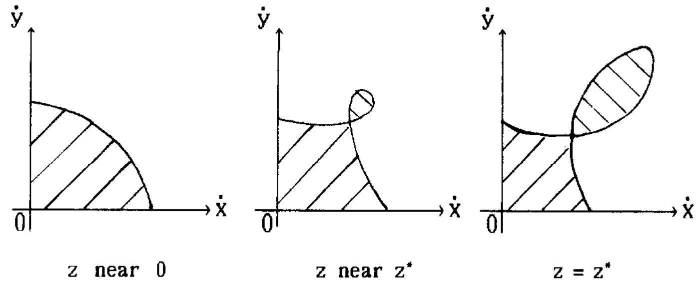

FIGURE 5

This theorem has a geometric interpretation. By the definition of $r(z)$ and by Lemma 3.3 we can write the conditions above as follows

$$\alpha'(z) \neq 0, \qquad r(z) = 0, \qquad r'(z) \neq 0;$$

up to the factor $(1/2\alpha f(z))\, r(z)$ is equal to the area of the sector in $\mathbb{R}^2$ which is cut out by the curve $(\dot{x}(\cdot, z), \dot{y}(\cdot, z))$.

Hence we can formulate our conditions geometrically (see Fig. 5):

(1) The primary branch of f (or more precisely: the projection of $\mathrm{PB}(f) \subseteq \mathbb{R} \times C_0$ under $\mathbb{R} \times C_0 \ni (\alpha, \varphi) \mapsto (\alpha, \varphi(1)) \in \mathbb{R} \times \mathbb{R}$ onto $\mathbb{R} \times \mathbb{R}$) has a nonvertical tangent at z,

(2) the area of the sector which is cut out by $(\dot{x}(\cdot, z), \dot{y}(\cdot, z))$ vanishes and

(3) is passed with a nonzero speed.

4. APPLICATIONS

Now we shall show, that $f(x) = -x/(1 + x^2)$, $x \in \mathbb{R}$, satisfies the assumptions of Theorem 3.1 and hence possesses a differentiable bifurcation point. Let $z > 0$ and $\alpha(z)$, $x(\cdot, z)$, $y(\cdot, z)$ be as in Theorem 1.1. Define

$$I(z) := \int_0^1 [f'(x)\, f^2(y) + f'(y)\, f^2(x)](t, z)\, dt$$

$$J(z) := \int_0^1 [f'(x)\, f'(y)](t, z)\, dt.$$

Then $r(z) = \frac{1}{2}(\alpha^2(z)/f(z))\, I(z)$, and we have to show:

There is a $z^* > 0$ *such that* $\alpha'(z^*) \neq 0$, $I(z^*) = 0$, $J(z^*) \neq 0$.

To that end we need some preliminary remarks, which we shall not prove.

(1) Let $0 \leqslant z \leqslant \sqrt{3}$, u, $v \geqslant 0$ and $uv + u + v = z^2$. Then $3uv \leqslant z^2$. (Since $v = (z^2 - u)/(1+u)$, $z^2 - 3uv = 1/(1+u)[z^2(3-z^2) + (3u - z^2)^2] \geqslant 0$).

(2) $f'(x) = (x^2-1)/(1+x^2)^2$ and $F(x) := \int_0^x f(u)\,du = -\frac{1}{2}\ln(1+x^2)$.

(3) Since $F(x(t,z)) + F(y(t,z)) = F(z)$ for all $t \in \mathbb{R}$ we have

$$(1+x^2)(1+y^2) = 1 + z^2 \quad \text{resp.} \quad x^2y^2 + x^2 + y^2 = z^2.$$

(4) Let $z > 0$ and $g(u,v) := (1/(1+z^2)^2)(3u^2v^2 - z^2)$ for $u, v \in \mathbb{R}$. From (3) we derive for $t \in \mathbb{R}$, $u = x(t,z)$, $v = y(t,z)$:

$$f'(v)\, f^2(u) + f'(u)\, f^2(v) = g(u,v),$$

hence $I(z) = \int_0^1 g(x(t,z), y(t,z))\,dt$.

(5) Let $z > 0$ and $h(u,v) := (1/(1+z^2)^2)(2u^2v^2 + 1 - z^2)$ for $u, v \in \mathbb{R}$. Again (3) yields for $t \in \mathbb{R}$, $u = x(t,z)$, $v = y(t,z)$:

$$f'(u)\, f'(v) = h(u,v),$$

and hence $J(z) = \int_0^1 h(x(t,z), y(t,z))\,dt$.

(6) $2g(u,v) - 3h(u,v) = (z^2-3)/(1+z^2)^2$ for all $u, v \in \mathbb{R}$.

(7) We have $I(z) < 0$ for $0 < z \leqslant \sqrt{3}$: For $t \in \mathbb{R}$, $u := x^2(t,z)$, $v := y^2(t,z)$ (3) gives $uv + u + v = z^2$. From (1) we find $g(x(t,z), y(t,z)) = (1/(1+z^2)^2)(3uv - z^2) \leqslant 0$; and thus

$$I(z) = \int_0^1 g(x(t,z), y(t,z))\,dt \leqslant 0.$$

Since $g(x(0,z), y(0,z)) = g(0, -z) = -z^2/(1+z^2)^2 < 0$ we see $I(z) < 0$ for $0 < z \leqslant \sqrt{3}$.

(8) $2I(z) - 3J(z) = \int_0^1 [2g - 3h](x(t,z), y(t,z))\,dt = (z^2-3)/(1+z^2)^2$.

(9) Suppose $I(z) = 0$ for some $z > 0$. Then $J(z) \neq 0$, since otherwise (8) implies $z = \sqrt{3}$, a contradiction to (7).

It remains to prove

(i) $\alpha'(z) > 0$ for all $z > 0$,

(ii) there is a $z > 0$ with $I(z) > 0$,

since then (7) yields a $z^* > \sqrt{3}$ with $I(z^*) = 0$, and (i) and (9) show that z^* is a bifurcation point.

For this purpose we need some knowledge on complete elliptic integrals:

(10) For $m < 1$ define

$$E(m) := \int_0^{\pi/2} (1 - m\sin^2\varphi)^{1/2}\,d\varphi, \qquad K(m) := \int_0^{\pi/2} \frac{1}{(1 - m\sin^2\varphi)^{1/2}}\,d\varphi.$$

Note

$$K'(m)=\frac{1}{2}\int_0^{\pi/2}\frac{\sin^2\varphi}{(1-m\sin^2\varphi)^{3/2}}\,d\varphi.$$

(11) For $m\to 1-$ the asymptotic formulae hold (cmp. formula 112.01 in [2]):

$$\lim_{m\to 1-} E(m)=1, \qquad \lim_{m\to 1-}\left[K(m)+\frac{1}{2}\ln(1-m)\right]=\ln 4,$$

$$\lim_{m\to 1-}\left[K'(m)-\frac{1}{2}\frac{1}{1-m}\right]=0.$$

(12) From formula 17.4.17 in [1] we derive

$$K(m)=\frac{1}{\sqrt{1-m}}K\left(\frac{m}{m-1}\right), \qquad E(m)=\sqrt{1-m}\,E\left(\frac{m}{m-1}\right)$$

$$K'(m)=\frac{1}{2}\frac{1}{(1-m)^{3/2}}K\left(\frac{m}{m-1}\right)-\frac{1}{(1-m)^{5/2}}K'\left(\frac{m}{m-1}\right).$$

(13) Let $z>0$ and $t\in[0,1]$, hence $y(t,z)\leqslant 0$. Then (3) gives $y=(z^2-x^2)/(1+x^2)$ and

$$f(y)=-\frac{y}{1+y^2}=\frac{|y|}{1+y^2}=\frac{1+x^2}{1+z^2}\sqrt{\frac{z^2-x^2}{1+x^2}}=\frac{1}{1+z^2}\sqrt{z^2-x^2}\sqrt{1+x^2}.$$

Proof of (i). Let $x(t,z)=z\sin\varphi(t,z)$ with $\varphi(0,z)=0$, $\varphi(1,z)=\pi/2$. Then

$$z\cos\varphi\dot{\varphi}=\dot{x}=\alpha f(y)=\frac{\alpha}{1+z^2}z\cos\varphi\sqrt{1+z^2\sin^2\varphi}$$

resp.

$$\dot{\varphi}=\frac{\alpha}{1+z^2}\sqrt{1+z^2\sin^2\varphi}.$$

Thus we have for $t\in[0,1]$:

$$\alpha t=(1+z^2)\int_0^{\varphi(t,z)}\frac{d\varphi}{\sqrt{1+z^2\sin^2\varphi}}$$

resp.

$$\alpha(z)=(1+z^2)\,K(-z^2).$$

Differentiation yields

$$\alpha'(z) = z \int_0^{\pi/2} \frac{1 + \cos^2\varphi + z^2 \sin^2\varphi}{(1 + z^2 \sin^2\varphi)^{3/2}}\, d\varphi;$$

this proves $\alpha'(0) = 0$ and $\alpha'(z) > 0$ for $z > 0$.

Remark 4.1. Furthermore we find $\alpha''(0) = \int_0^{\pi/2} (1 + \cos^2\varphi)\, d\varphi = \frac{3}{4}\pi = (\pi/8)(f'''(0)/[f'(0)]^2)$ (compare the notes behind Lemma 1.7).

Remark 4.2. If $m = -z^2$ then (12) yields $\alpha(z) = \sqrt{1 + z^2}\, K(z^2/(1 + z^2))$, and from (11) we find the asymptotic expression

$$\lim_{z \to \infty} [\alpha(z) - z \cdot \ln(4z)] = 0.$$

Proof of (ii). From (2) and (13) we derive

$$\begin{aligned} I(z) &= 2\frac{f(z)}{\alpha^2(z)}\, r(z) = 2\int_0^1 f^2(y(t, z))\, f'(x(t, z))\, dt \\ &= \frac{2}{\alpha}\int_0^1 f(y)\, f'(x)\dot{x}\, dt \\ &= \frac{2}{\alpha}\int_{x(0,z)}^{x(1,z)} \frac{1}{1 + z^2}\sqrt{z^2 - x^2}\,\sqrt{1 + x^2}\,\frac{x^2 - 1}{(1 + x^2)^2}\, dx \\ &= \frac{2}{\alpha}\frac{1}{1 + z^2}\int_0^z \sqrt{z^2 - x^2}\,\frac{x^2 - 1}{(1 + x^2)^{3/2}}\, dx, \end{aligned}$$

and the substitution $x = z \sin\varphi$, $dx = z\cos\varphi\, d\varphi$ gives

$$\begin{aligned} I(z) &= \frac{2}{\alpha(z)}\frac{z^2}{1 + z^2}\int_0^{\pi/2} \cos^2\varphi\,\frac{z^2\sin^2\varphi - 1}{(1 + z^2\sin^2\varphi)^{3/2}}\, d\varphi \\ &= \frac{2}{\alpha(z)}\frac{z^2}{1 + z^2}\int_0^{\pi/2} \frac{-1 + (1 + z^2)\sin^2\varphi - z^2\sin^4\varphi}{(1 + z^2\sin^2\varphi)^{3/2}}\, d\varphi \\ &= \frac{2}{\alpha(z)}\frac{1}{1 + z^2} \\ &\quad \times \int_0^{\pi/2} \frac{-(1 + z^2\sin^2\varphi)^2 + (1 - z^2)(1 + z^2\sin^2\varphi) + 2z^2(1 + z^2)\sin^2\varphi}{(1 + z^2\sin^2\varphi)^{3/2}}\, d\varphi \\ &= \frac{2}{\alpha(z)}\frac{1}{1 + z^2}[-E(-z^2) + (1 - z^2)K(-z^2) + 4z^2(1 + z^2)K'(-z^2)]. \end{aligned}$$

Let $m := -z^2$, $n := m/(m-1) = z^2/(1+z^2)$, then $1-n = 1/(1+z^2)$, $(1-n)(1-m) = 1$ and Remark 4.2 imply $(1+z^2)\,\alpha(z) = (1+z^2)^{3/2} K(n) = K(n)/(1-n)^{3/2}$. Now (12) yields

$$\begin{aligned}
I(z) &= \frac{2}{K(n)}(1-n)^{3/2}\,[-E(m) + (1+m)\,K(m) - 4m(1-m)\,K'(m)]\\
&= \frac{2}{K(n)}(1-n)^{3/2}\Bigg[-\sqrt{1-m}\,E(n) + \frac{1+m}{\sqrt{1-m}}\,K(n)\\
&\quad - 4m(1-m)\cdot\frac{1}{2}\cdot\frac{1}{(1-m)^{3/2}}\,K(n) + 4m(1-m)\cdot\frac{1}{(1-m)^{5/2}}\,K'(n)\\
&= \frac{2}{K(n)}(1-n)^{3/2}\left[-\sqrt{1-m}\,E(n) + \sqrt{1-m}\,K(n) + \frac{4m}{(1-m)^{3/2}}\,K'(n)\right]\\
&= \frac{2}{K(n)}(1-n)\sqrt{1-n}\,\sqrt{1-m}\left[-E(n) + K(n) + \frac{m}{1-m}\,\frac{4}{1-m}\,K'(n)\right]\\
&= \frac{2}{K(n)}(1-n)[K(n) - E(n) - n\cdot 4\cdot(1-n)\,K'(n)].
\end{aligned}$$

Hence we have to prove that $\hat{I}(z) := K(n) - E(n) - 4n(1-n)\,K'(n)$, $n = n(z) = z^2/(1+z^2)$, admits positive values. Since $\lim_{z\to\infty} n(z) = 1$, we replace K, E, K' by the asymptotic expressions of (11) and obtain

$$\begin{aligned}
\hat{I}(z) &\sim \ln 4 - \frac{1}{2}\ln(1-n) - 1 - 4\cdot 1\cdot(1-n)\cdot\frac{1}{2}\frac{1}{1-n}\\
&= \ln\left(\frac{4}{\sqrt{1-n}}\right) - 3 = \ln(4\sqrt{1+z^2}) - 3 \sim \ln(4z) - 3,
\end{aligned}$$

resp.

$$\lim_{z\to\infty}\,(\hat{I}(z) - \ln(4z) + 3) = 0.$$

It follows that $\lim_{z\to\infty} \hat{I}(z) = \infty$, which proves (ii). Moreover we find that the zeros of $I(z)$ are bounded from above. It is easy to evaluate the integral

$$\int_0^{\pi/2} \cos^2\varphi\,\frac{z^2\sin^2\varphi - 1}{(1+z^2\sin^2\varphi)^{3/2}}\,d\varphi$$

numerically. We find $I(z^*) = 0$ for $z^* \approx 4.2536$. Moreover, z^* is (numerically) the only bifurcation point of f.

Final Remarks

(1) Let $f_c(x) := (1/c)\, f(cx) = -x/(1+c^2x^2)$ for $c>0$, $x \in \mathbb{R}$. Observe, that if x is a solution of (αf), then $x_c := (1/c)x$ is a solution of (αf_c). Thus $z_c^* := (1/c)z^*$ is a differentiable bifurcation point of f_c.

(2) Suppose that f is decreasing on $[0, x^*]$, i.e., $f'(x) \leqslant 0$ for $0 \leqslant x \leqslant x^*$. Then Theorem 3.1 and $f'(0)<0$ imply $r(z)<0$ for $z \in [0, x^*]$. Hence x^* is a lower bound for all bifurcation points z^* of f. In particular, there is no bifurcation point if f has no relative minimum on $\mathbb{R}^+$. In our example $f(x) = -x/(1+x^2)$ we have $x^* = 1$.

(3) Let $\omega \in \mathbb{R}$, $\alpha \in \mathbb{R}$, $x \in C(\mathbb{R})$ and $k \in \mathbb{N}$ such that $k\omega + 2 \neq 0$. Define

$$\omega_k := \frac{2\omega}{k\omega+2}, \qquad \alpha_k := (-1)^k \left(1+\frac{k}{2}\omega\right)\alpha,$$

$$x_k(t) := (-1)^k\, x\left(\left(1+\frac{k}{2}\omega\right)t\right).$$

Further let $f \in C(\mathbb{R})$, $f(0)=0$, $f(x) \neq 0$ for $x \neq 0$, and suppose that $x\colon \mathbb{R} \to \mathbb{R}$ is a periodic solution of (αf) with period ω and amplitude $z := x(1)$. Assume $x(0)=0$ but $x \not\equiv 0$. Then

(i) if k is even, x_k is a periodic solution of $(\alpha_k f)$ with period ω_k, and we have $x_k(0)=0$, $x_k(1)=z$,

(ii) if x is symmetric, then x_k is a symmetric periodic solution of $(\alpha_k f)$ for all $k \in \mathbb{N}$.

This group of transformations was given by Saupe [14]. For our example $f(x) = -x/(1+x^2)$ we thus have the bifurcation diagram of Fig. 6. Since

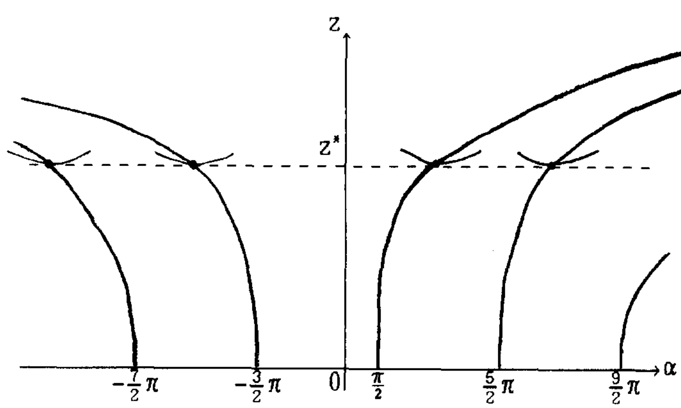

Figure 6

the transformation for $k=1$ leaves the Kaplan–Yorke branch pointwise fixed, we are able to study the bifurcation point more precisely. One finds (compare [14]):

The bifurcating branch can be parameterized near the bifurcation point by the period ω *or by the delay parameter* α.

Parameterization by ω *yields*

$$\alpha(4)=\alpha^*, \qquad z(4)=z^*, \qquad \frac{d\alpha}{d\omega}(4)=-\frac{1}{4}\alpha^*, \qquad \frac{dz}{d\omega}(4)=0,$$

Parameterization by α *yields*

$$\omega(\alpha^*)=4, \qquad z(\alpha^*)=z^*, \qquad \frac{d\omega}{d\alpha}(\alpha^*)=-\frac{4}{\alpha^*}, \qquad \frac{dz}{d\alpha}(\alpha^*)=0.$$

It follows that the bifurcating branch has a horizontal tangent at the bifurcation point, the periods on the right-hand side of α^* are less than 4, on the left-hand side greater than 4 (compare numerical results of Hadeler [6]).

(4) Suppose that $f\in C(\mathbb{R})$ is Lipschitzian and has only $0\in\mathbb{R}$ as a zero. Let $x\colon \mathbb{R}\to\mathbb{R}$ be a periodic solution with $x(0)=0$ but $x\not\equiv 0$. If ω is the period of x then $\omega\notin\{2/n \mid n\in\mathbb{Z}\setminus\{0\}\}=\{2, 1, \frac{2}{3}, \frac{1}{2}, ...\}$.

Proof. If $\omega=1$, then $x(t-1)=x(t)$, $x(0)=0$ and $\dot{x}(t)=\alpha f(x(t-1))=\alpha f(x(t))$ imply $x\equiv 0$.

Assume $\omega=2$, i.e., $x(\cdot+2)=x$. Define $z:=x(1)$, $y:=x(\cdot-1)$; then

$$\begin{aligned} &\dot{x}=\alpha f(y), \qquad x(0)=0, \qquad && x(1)=z \\ &\dot{y}=\alpha f(x), \qquad y(0)=x(-1)=x(1)=z, \qquad && y(1)=0. \end{aligned} \tag{$*$}$$

Let $F:=\int_0^{\cdot} f(u)\,du$; then $t\mapsto F(x(t))-F(y(t))$ is constant and thus we have $-F(z)=F(x(0))-F(y(0))=F(x(1))-F(y(1))=F(z)$, or $F(z)=0=F(0)$. If $z\neq 0$, e.g., $z>0$, then $f(u)=F'(u)=0$ for some $u\in\,]0, z[$, a contradiction to $f(u)\neq 0$ for $u\neq 0$. Hence $z=0$, and $(*)$ gives $x=y=0$. Suppose now $\omega=2/n$ for some $n\in\mathbb{Z}\setminus\{0\}$. Then an appropriate transformation produces a periodic solution with period 1 or 2. ∎

Results of this type can also be found in [12].

(5) Let $\mathbb{B}$ be a Banachspace and $\mathfrak{F}\colon \mathbb{R}\times\mathbb{B}\to\mathbb{R}$ a C^2-map. For $b\in\mathbb{B}$ let $f_b:=\mathfrak{F}(\cdot, b)\colon \mathbb{R}\to\mathbb{R}$ and for $x\in\mathbb{R}$ suppose

$$f_b(-x)=-f_b(x), \qquad xf_b(x)<0 \quad \text{if} \quad x\neq 0, \qquad f_b'(0)<0.$$

Furthermore let $(x, y)\colon \mathbb{R}\times\mathbb{R}\times\mathbb{B}\to\mathbb{R}^2$ and $\alpha\colon \mathbb{R}\times\mathbb{B}\to\mathbb{R}$ be the maps which are defined by

$$\dot{x}(t, z, b)=\alpha(z, b)\,\mathfrak{F}(y(t, z, b), b),\qquad x(0, z, b)=0,\qquad x(1, z, b)=z$$
$$\dot{y}(t, z, b)=-\alpha(z, b)\,\mathfrak{F}(x(t, z, b), b),\quad y(0, z, b)=-z,\quad y(1, z, b)=0$$

for $(t, z, b)\in\mathbb{R}\times\mathbb{R}\times\mathbb{B}$. Then $x(\cdot, z, b)$ is a special periodic solution of $(\alpha(z, b) f_b)$ for all $(z, b)\in\mathbb{R}\times\mathbb{B}$. Finally define

$$r(z, b):=\frac{\alpha^2(z, b)}{\mathfrak{F}(z, b)}\int_0^1 \mathfrak{F}^2(x(t, z, b), b)\,\mathfrak{F}_x(y(t, z, b), b)\,dt$$

for $(z, b)\in\mathbb{R}\times\mathbb{B}$, and suppose

$$\alpha_z(z^*, b^*)\neq 0,\qquad r(z^*, b^*)=0,\qquad r_z(z^*, b^*)\neq 0.$$

for some $(z^*, b^*)\in\mathbb{R}\times\mathbb{B}$. Then z^* is a differentiable bifurcation point of $f^*:=\mathfrak{F}(\cdot, b^*)$, and the implicit function theorem yields an open neighbourhood $U\subseteq\mathbb{B}$ of $b^*\in\mathbb{B}$ and a differentiable map $z\colon U\to\mathbb{R}$ such that

$$z(b^*)=z^*,\qquad r(z(b), b)=0,$$
$$r_z(z(b), b)\neq 0,\qquad \alpha_z(z(b), b)\neq 0,$$

It follows that $z(b)$ is a differentiable bifurcation point of f_b for all $b\in U$. Let us express this fact in the following way: *Differentiable bifurcation of symmetric periodic solutions is an "open" and "differentiable" property.*

EXAMPLE. $\mathbb{B}=\mathbb{R}$, $\mathfrak{F}(x, b)=-x/(1+|x|^b)$, $b^*=2$ (see also [14]).

(6) Let $g\in C^2(\mathbb{R}^{n+1})$, $n\in\mathbb{N}_0$, and define $f(x):=g(x, -x, x, \ldots, \pm x)$ for $x\in\mathbb{R}$. Suppose that f is odd and let $(x(\cdot, z), y(\cdot, z))\colon \mathbb{R}\to\mathbb{R}^2$, $\alpha(z)$ for $z>0$ be as in Theorem 1.1. Then $x(\cdot, z)$ is not only a symmetric periodic solution of $(\alpha(z) f)$, but also of

$$\dot{x}(t)=\alpha g(x(t-1), x(t-3), \ldots, x(t-2n-1)). \tag{αg}$$

Hence this equation has a primary branch of special periodic solutions too, and we can study analogously our bifurcation problem for symmetric periodic solutions. In the following we only state the main results. The important Lemma 2.2 has the following analogue:

$\lambda \in \mathbb{R}\setminus\{0\}$ is an eigenvalue of $W(z)$ if and only if there is a solution (η, ζ) of

$$\dot{\eta} = \alpha(z)\, h(\lambda, x(\cdot, z)), \qquad \eta(0) = \zeta(1)$$
$$\lambda\dot{\zeta} = \alpha(z)\, h(\lambda, y(\cdot, z)), \qquad \lambda\zeta(0) = \eta(1),$$

where

$$h(\lambda, x) := f_1(x) + \frac{1}{\lambda} f_2(x) + \cdots + \frac{1}{\lambda^n} f_{n+1}(x),$$

$$f_k(x) := \frac{\partial g}{\partial x_k}(x, -x, ...)$$

for $x \in \mathbb{R}$, $k = 1, ..., n+1$.

The bifurcation theorem for (αg) can be stated as follows:

Let $g \in C^2(\mathbb{R}^{n+1})$, $\quad n \in \mathbb{N}_0$ *and for* $x \in \mathbb{R}$ *define*

$$f(x) := g(x, -x, ..., \pm x),$$

$$b(x) := \left(\frac{\partial g}{\partial x_1} - 2\frac{\partial g}{\partial x_2} + 3\frac{\partial g}{\partial x_3} - \cdots\right)(x, -x, ..., \pm x).$$

Suppose that f *is odd and* $xf(x) < 0$ *for* $x \neq 0$. *Let* $(x, y): \mathbb{R} \to \mathbb{R}^2$ *be the Kaplan–Yorke solution of* $(\alpha(z) f)$ *with amplitude* $z > 0$. *Suppose that*

(1) $\alpha'(z) \neq 0$,
(2) $\int_0^1 [f^2(x)\, b(y) + f^2(y)\, b(x)]\, dt = 0$,
(3) $\int_0^1 [f'(x)\, b(y) + f'(y)\, b(x)]\, dt \neq 0$,

then z *is a differentiable bifurcation point of* (g) *where symmetric periodic solutions bifurcate.*

Let $f(x) = -x/(1+x^2)$. Then one can easily construct an example that fulfils these conditions. For example define

$$g(x_1, ..., x_{n+1}) := f(a_1 x_1 + \cdots + a_{n+1} x_{n+1}),$$

where $a_1 - a_2 + a_3 - \cdots = 1$, $a := a_1 - 2a_2 + 3a_3 - \cdots \neq 0$. Then $b = af'$, and g possesses a bifurcation point. Observe that (f) and (g) have the same Kaplan–Yorke solutions, but the bifurcation solutions need not necessarily be the same.

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